Rebalancing an Investment Portfolio in the Presence of Convex Transaction Costs and Market Impact Costs

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Dedicated to Kees Roos on the occasion of his 70th birthday, in recognition of his myriad contributions to conic programming and interior point methods.

Abstract

The inclusion of transaction costs is an essential element of any realistic portfolio optimization. In this paper, we extend the standard portfolio problem to consider convex transaction costs that are incurred to rebalance an investment portfolio. Market impact costs measure the effect on the price of a security that result from an effort to buy or sell the security, and they can constitute a large part of the total transaction costs. The loss to a portfolio from market impact costs is typically modeled with a convex function that can usually be expressed using second order cone constraints. The Markowitz framework of mean-variance efficiency is used. In order to properly represent the variance of the resulting portfolio, we suggest rescaling by the funds available after paying the transaction costs. This results in a fractional programming problem, which can be reformulated as an equivalent convex program of size comparable to the model without transaction costs. An optimal solution to the convex program can always be found that does not discard assets. The results of the paper extend the classical Markowitz model to the case of convex transaction costs in a natural manner with limited computational cost.
Keywords: Portfolio optimization, transaction costs, market impact costs, rebalancing, conic optimization, convex optimization.

1 Introduction

Constructing a portfolio of investments is one of the most significant financial decisions facing individuals and institutions. A decision-making process must be developed which identifies the appropriate weight each investment should have within the portfolio. The portfolio must strike what the investor believes to be an acceptable balance between risk and reward. In addition, the costs incurred when setting up a new portfolio or rebalancing an existing portfolio must be included in any realistic analysis. In this paper, we consider convex transaction costs, including linear (proportional) transaction costs, piecewise linear transaction costs, quadratic transaction costs, and market impact costs. In order to properly reflect the effect of transaction costs, we suggest rescaling the risk term by the funds available after paying the transaction costs.

Essentially, the standard portfolio optimization problem is to identify the optimal allocation of limited resources among a limited set of investments. Optimality is measured using a tradeoff between perceived risk and expected return. Expected future returns are based on historical data. Risk is measured by the variance of those historical returns.

When more then one investment is involved, the covariance among individual investments becomes important. In fact, any deviation from perfect positive correlation allows a beneficial diversified portfolio to be constructed. Efficient portfolios are allocations that achieve the highest possible return for a given level of risk. Alternatively, efficient portfolios can be said to minimize the risk for a given level of return. These ideas earned their inventor a Nobel Prize and have gained such wide acceptance that countless references could be cited; however, the original source is Markowitz [33].

One standard formulation of the portfolio problem minimizes a quadratic risk measurement with a set of linear constraints specifying the minimum expected portfolio return, E_0, and enforcing full investment of funds. The decision variables x_i are the proportional weights of the i^{th} security in the portfolio. Here n securities are under consideration. Let μ_i denote the expected value of \$1 invested in security i at the end of the period of interest, and let Q denote the positive semidefinite covariance matrix of these values. When short selling is not allowed, the proportions x_i are restricted to be nonnegative and the resulting formulation is:

$$\begin{array}{lrl} \min & \frac{1}{2}x^TQx & \\ \text{s.t.} & \mu^Tx & \geq E_0 \\ & e^Tx & = 1 \\ & x & \geq 0, \end{array} \tag{1}$$

where e denotes the vector of all ones. By varying the parameter E_0 and solving multiple instances of this problem, the set of efficient portfolios can be generated. This set, visualized in a risk/return plot, is called the efficient frontier. An investor may decide where along the efficient frontier (s)he finds an acceptable balance between risk and reward.

In this paper, we describe two methods for finding an optimal portfolio when convex transaction costs have to be paid. Both methods require the solution of a convex program of similar size to the Markowitz model. The models allow different costs for different securities, and different costs for buying and selling, and capture the feature that transaction costs are paid when a security is bought or sold and the transaction cost reduces the amount of that particular security that is available. The first model is related to a model developed by Lobo et al. [31]. The second model is a refinement where the risk measure is scaled to reflect the loss of principal due to transaction costs. This rescaled objective can be regarded as the risk per dollar invested, so both the risk and the return in our model are measured using the portfolio arising after paying the transaction costs. We show that various types of transaction costs lead to conic optimization problems in both models. We show that the first model may lead to the discarding of assets if a complementarity constraint is not imposed between buy and sell decisions, because discarding assets can reduce the measure of risk. In contrast, an optimal solution to the model with the rescaled objective will not result in assets being discarded.

Chen et al. [15] survey research on the incorporation of transaction costs into the Markowitz model. The portfolio rebalancing problem has similarities to the index tracking problem [1, 13, 18]. See Zenios [49] for a discussion of portfolio optimization models. The optimal solution to the portfolio optimization problem is sensitive to the data Q and μ, so estimating this data accurately is the subject of current research; see Chopra and Ziemba [16] or Bengtsson and Holst [5] for example. Stochastic programming approaches to portfolio optimization have been investigated in [17, 25, 38, 39] and elsewhere; such approaches work with sets of scenarios.

Modification of a portfolio should be performed at regular intervals, and determination of the appropriate interval in the presence of transaction costs is a problem of interest. Preferably, selection of the interval should be done in conjunction with selection of the method used for rebalancing. This paper contains a method for rebalancing. There has been interest in portfolios that can be modified continuously, starting with Merton [35]. These methods are generally limited computationally to problems with a small number of securities. For a survey on the impact of transaction costs on the dynamic rebalancing problem, see Cadenillas [11]. For a discussion of handling capital gains taxes in dynamic portfolio allocation problems, see Cadenillas and Pliska [12].

This paper is organized as follows. We turn to the portfolio rebalancing problem in §2. First, we motivate the cost model and provide examples of costs that fit this model, including market impact costs. We give an example where the unscaled model gives a solution that is not an attractive practical strategy. We show that this particular type of solution cannot arise if there is a riskless asset. Alternative approaches to this problem are also discussed in §2. A scaling method for preventing the unattractive solution is presented in §3, with the properties of the optimal solution returned by this method given in §4. This method leads to a model which can be cast as a quadratic programming problem, with additional convex constraints depending on the structure of the transaction costs. Several types of costs are considered in §4, including market impact costs that lead to second order cone programs. Efficiently finding a portfolio to maximize the Sharpe ratio is the subject of §5. Computational results for two empirical datasets are presented in §6. Finally, we offer concluding remarks in §7.

2 Portfolio Rebalancing Problem

What we consider is an extension of the basic portfolio optimization problem in which transaction costs are incurred to rebalance a portfolio, $\bar{x}$, into a new and efficient portfolio, x. A portfolio may need to be rebalanced periodically simply as updated risk and return information is generated with the passage of time. Further, any alteration to the set of investment choices would necessitate a rebalancing decision of this type.

In addition to the obvious cost of brokerage fees/commissions, here are two examples of other transaction costs that can be modeled in this way:

1. Capital gains taxes are a security-specific selling cost that can be a major consideration for the rebalancing of a portfolio. For more discussion of the impact of capital gains, especially in a dynamic portfolio allocation model, see Cadenillas and Pliska [12].

2. Another possibility would be to incorporate an investor's confidence in the risk/return forecast as a subjective "cost". Placing high buying and selling costs on a security would favor maintaining the current allocation $\bar{x}$. Placing a high selling cost and low buying cost could be used to express optimism that a security may outperform its forecast.

The transaction costs are assumed to be convex. To obtain a portfolio x from an initial portfolio $\bar{x}$, we pay transaction costs $c(x-\bar{x})$, where c is a convex nonnegative function with $c(0) = 0$. Transaction costs which meet this model include proportional, convex piecewise linear, and convex quadratic transaction costs. Proportional transaction costs have the structure:

$$c(x-\bar{x}) = \sum_{i=1}^{n} c_i(x_i - \bar{x}_i) \text{ where } c_i(x_i - \bar{x}_i) = \begin{cases} c_i^B(x_i - \bar{x}_i) & \text{if } x_i > \bar{x}_i \\ c_i^S(\bar{x}_i - x_i) & \text{otherwise} \end{cases} \tag{2}$$

where c_i^B and c_i^S are positive constants.

We normalize so that the initial wealth is $e^T\bar{x} = 1$. The resulting model for minimizing the variance of the resulting portfolio subject to meeting an expected return of $E_0 > 0$ in the presence of transaction costs is

$$\begin{array}{lrcl} \min & \frac{1}{2}x^TQx & & \\ \text{s.t.} & \mu^T x & \geq & E_0 \\ & e^Tx + c(x-\bar{x}) & = & 1 \end{array} \tag{3}$$

Note that we are allowing short selling here. If transaction costs are linear and if short selling costs are also linear and at least as large as the cost of selling the asset, we can model the transaction costs for security i as the convex piecewise linear function

$$c_i(x_i - \bar{x}_i) = \begin{cases} c_i^B(x_i - \bar{x}_i) & \text{if } x_i > \bar{x}_i \\ c_i^S(\bar{x}_i - x_i) & \text{if } 0 \leq x_i \leq \bar{x}_i \\ c_i^S\bar{x}_i - c_i^{SS}x_i & \text{if } x_i < 0 \end{cases} \tag{4}$$

where c_i^{SS} is the proportional cost for short selling, with $c_i^{SS} \geq c_i^S$. More general piecewise linear convex transaction cost functions were considered by Potaptchik et al. [41] and by Bertsimas et al. [6], for example.

An important part of the cost of trading a security is the market impact cost; it may be the major part of the total transaction cost [46]. The market impact cost measures the impact of trading a security that is somewhat illiquid on the price of a security. As more of a security is bought or sold, the proportional cost increases due to the scarcity effect. Let $f(v)$ denote the change in price due to a transaction of size v. To simplify the presentation, we assume v is positive; the treatment of negative v is analogous. It is generally accepted that $f(v)$ is a concave monotonically increasing function [20]. Torre [46] showed that $f(v) \propto \sqrt{v}$ fits the data well, a currently popular choice [19, 37] that "has been widely used in practice for many years" [19]. Alternative choices have been proposed. For example, Almgren et al. [3] determined empirically that $f(v) \propto v^{0.6}$ for some securities, and Zagst and Kalin [48] have used a piecewise linear function. Assuming the security is purchased at the modified cost, the overall market impact cost is $vf(v)$, a convex function in $v > 0$ when $f(v) \propto v^\gamma$ for any nonnegative γ. If the transaction can be split into several pieces then the market impact cost may be reduced; such models are considered by Moro et al. [37], for example.

Our assumptions on the transaction cost function are listed below:

Assumption 1 *The transaction cost function c satisfies the following:*

1. *$c(x)$ is a convex function of x.*
2. *$c(0) = 0$.*
3. *$c(x) \geq 0$ for all x.*

Continuous nonconvex transaction cost functions are considered in §4.4.

The return on an asset is the proportional increase in the value of the asset from the beginning to the end of the period. We represent the expected return using ρ, so $\rho = \mu - e$. When there are no transaction costs we have $e^T x = 1$, so the constraint $\mu^T x \geq E_0$ is equivalent to the constraint $\rho^T x \geq E_0 - 1$. However, when transaction costs are incurred, we will have $\mu^T x < \rho^T x + 1$ for asset allocations x, and so in this paper we use the more restrictive constraint $\mu^T x \geq E_0$.

A user might also require restrictions such as limiting the proportion of assets that can be invested in a group of securities. We can express this as a homogeneous constraint on x. For example, if security 1 must constitute no more than 10% of the resulting portfolio, we can impose the constraint

$$9x_1 - \sum_{i=2}^{n} x_i \leq 0.$$

We generalize this to allow m homogeneous constraints in our model, written in the form $Ax \leq 0$ where A is an $m \times n$ matrix and a is an m-vector. Note that a restriction on short selling can be modelled using a constraint of this form.

Model (3) contains a nonlinear equality constraint, so it is not a convex optimization problem. It can be made convex by relaxing the transaction cost constraint, giving the problem:

$$\begin{array}{lrcl} \min & \frac{1}{2}x^T Q x & & \\ \text{s.t.} & \mu^T x & \geq & E_0 \\ & e^T x + c(x - \bar{x}) & \leq & 1 \\ & Ax & \leq & 0 \end{array} \tag{5}$$

A drawback to this model is that it may result in an optimal solution where assets are discarded, as we show in §2.1. When the transaction costs are proportional as in (2), separate buy and sell variables can be introduced for each security and the equality form of the transaction cost constraint can be used. However, if a complementarity restriction on the buy and sell variables is not also imposed, the optimal solution may require both buying and selling a particular asset, which is clearly not an advisable practical strategy. The example of §2.1 illustrates this phenomenon. If there is a riskless asset available then the transaction cost constraint will hold at equality at the optimal solution to (5), as we show in §2.2. We introduce the following terminology for these two cases.

Definition 1 *A feasible asset allocation x is* **wasteful** *if $e^T x + c(x - \bar{x}) < 1$. Otherwise it is called* **frugal**.

Lobo et al. [31] give a model equivalent to (5) for the case of linear transaction costs. Their model maximizes the expected return, subject to the variance being below some threshold. For the example given in §2.1, this leads to an optimal solution where the transaction cost inequality is satisfied strictly, corresponding to discarding assets.

Adcock and Meade [1] add a linear term for the costs to the original Markowitz quadratic risk term and minimize this quantity. This requires finding an appropriate balance between the transaction costs and the risk. The model assumes a fixed rate of transaction costs across securities. The risk is measured in terms of the adjusted portfolio before transaction costs are paid. Konno and Wijayanayasake [29] consider a cost structure that is considerably more involved than ours, with the result that the model is harder to solve. Yoshimoto [47] considers a similar transaction cost model to ours and proposes a nonlinear programming algorithm to solve the problem and their computational results indicate that ignoring transaction costs can result in inefficient portfolios. In Braun and Mitchell [9], we investigated a different relaxation of (3) for the case of proportional transaction costs and showed that good solutions could be obtained using that relaxation. In particular, with such a cost structure, (3) can be reformulated as a quadratic program with complementarity constraints. The complementarity constraints impose a combinatorial structure on the problem and make it hard to solve. We examined a semidefinite programming relaxation of this problem in [9], and further references for this type of problem can be found there.

An alternative model is to reduce the vector of expected returns μ by the transaction costs. Best and Hlouskova [7] give an efficient solution procedure for this problem for the case of proportional transaction costs and a diagonal covariance matrix Q. The model implicitly assumes that transaction costs are paid at the end of the period, impacting both the risk and the return. If the transaction costs must be paid at the beginning of the period then care must be taken in the sale of assets to pay the transaction costs, in order to ensure that the resulting portfolio has securities in the same proportion. Further, the return calculation assumes a return on the amount paid in transaction costs, so this constraint needs to be modified.

If the transaction costs include a fixed cost per transaction, one modeling approach is to either place an upper bound on the number of transactions or to include a penalty term for the number of transactions. This gives rise to a mixed integer nonlinear programming problem. This approach has been investigated widely; see, for example, Perold [40], Bienstock [8], Bertsimas et al [6], Konno and Wijayanayasake [29], Kellerer et al. [27], and Lee and Mitchell [30]. The presence of the integrality restriction makes the formulation far harder to solve than the one presented in this paper.

2.1 An example with an unattractive optimal solution

In this section, we give an example where the optimal solution to (5) is wasteful. Assume we have two assets A and B, with expected end-of-period values $\mu_A = 1.5$ and $\mu_B = 1.05$. Assume all transaction costs are equal to 2%, so $c_B = c_S = 0.02e$. Assume the covariance matrix for the assets is

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

Assume wealth is initially equally split between the assets. Let the desired expected value be $E_0 = 1.1$.

With no transaction costs, the optimal solution to (1) is to take $x_A = 0.2308$ and $x_B = 0.7692$, giving an optimal value of 0.1154.

The optimal solution to the proportional transaction cost model (5) has $x_A = 0.2785$ and $x_B = 0.6498$, with an objective function value of 0.1021. This solution sells 0.2215 units of asset A and only buys 0.1498 units of asset B, and results in a slack value of 0.0643 in the transaction cost constraint. Thus, approximately 6.4% of the wealth is discarded by this wasteful solution. It is clearly not an attractive practical strategy. The solution to (3) where separate variables are introduced for buy and sell decisions but no complementarity restriction is imposed leads to the same solution.

The best frugal solution has $x_A = 0.2283$ and $x_B = 0.7610$, with value 0.1129 (all numbers given to four decimal places). This is a scaling of the optimal solution to the model with no transaction costs, for this example.

In §3, we suggest a modification of (5) which results in a convex optimization problem which always has a frugal optimal solution. For this example problem, the optimal solution to the modified problem is the frugal solution given above.

2.2 A risk-free security

In this section we consider the case that one of the available securities is riskless and incurs no transaction costs. We assume that the riskless security only appears in the additional linear constraints $Ax \leq 0$ through a nonnegativity restriction. We show that in such a situation the optimal solution to (5) satisfies the transaction cost constraint at equality. Thus in this case, it is not necessary to impose complementarity constraints between the buy and sell variables; or to rephrase, the mathematical program with complementarity constraints (MPCC) [32] that has explicit buy and sell variables can be solved by solving its convex relaxation (5).

Let $\mu_f > 1$ denote the value of the riskless security at the end of the period and let y denote the amount invested in this security. Problem (5) can be written equivalently as

$$\begin{array}{lcccl} \min & & & \frac{1}{2}x^T Q x & & \\ \text{s.t.} & \mu_f y & + & \mu^T x & \geq & E_0 \\ & y & + & e^T x + c(x - \bar{x}) & \leq & 1 \\ & & & Ax & \leq & 0 \\ & y & & & \geq & 0 \end{array} \tag{6}$$

where x, μ, c, and Q refer only to the risky securities. For any feasible solution with $y + e^T x + c(x - \bar{x}) < 1$, an equally good feasible solution with $y + e^T x + c(x - \bar{x}) = 1$ can be obtained by increasing y. Thus, there is an optimal solution to (6) with $y + e^T x + c(x - \bar{x}) = 1$. We can prove a stronger result.

Theorem 1 *Any optimal solution to (6) satisfies $y + e^T x + c(x - \bar{x}) = 1$ provided Q has full rank and $\mu > 0$.*

Proof: We prove this by contradiction. Let $\hat{x}$, $\hat{y}$ be a feasible solution to (6) with $\hat{y} + e^T \hat{x} + c(\hat{x} - \bar{x}) < 1$. We can construct a better feasible solution in two steps.

First, increase y to $\tilde{y} = \hat{y} + 0.5(1 - e^T \hat{x} - c(\hat{x} - \bar{x}) - \hat{y})$. The slack in the first constraint has been increased by $\mu_f(\tilde{y} - \hat{y})$, and the transaction cost constraint still has a positive slack. Thus, there exists a nonnegative $\alpha < 1$ such that $x = \tilde{x} := \alpha \hat{x}$ is feasible with $y = \tilde{y}$. Since $\alpha < 1$, it follows that $\tilde{x}^T Q \tilde{x} < \hat{x}^T Q \hat{x}$. Therefore, $\hat{x}$, $\hat{y}$ cannot be optimal for (6). □

3 Scaled Risk Measurement

To this point, we have been optimizing the standard risk measure for efficient frontiers, that is:

$$\frac{1}{2}x^T Q x.$$

When there are no transaction costs to be paid, one dollar is always available for investment, i.e. ($\sum_{i=1}^n x_i = 1$). This assumption is implicit in the standard risk measure. However, for nonzero transaction costs that implicit assumption is no longer valid: one dollar is not available for investment, costs will be paid to rebalance. The appropriate objective is therefore

$$f(x) := \frac{x^T Q x}{2(e^T x)^2}. \tag{7}$$

Here $1 - e^T x$ is the amount paid in transaction costs. Therefore $e^T x$ is the actual amount available for investment, so we are choosing to scale the standard risk measurement by the square of the dollar amount actually invested. The scaled objective is the risk per dollar invested.

This gives the fractional convex programming problem which we will solve to find the optimal portfolio for a given expected return.

$$\begin{array}{lccc} \min & \frac{x^T Q x}{2(e^T x)^2} & & \\ \text{s.t.} & \mu^T x & \geq & E_0 \\ & e^T x + c(x - \bar{x}) & \leq & 1 \\ & Ax & \leq & 0 \end{array} \tag{8}$$

As we will see, there is always a frugal optimal solution to this problem. This contrasts with the standard risk measure, which may have a wasteful optimal solution if there is no riskless security, as shown in §2.1.

Analytically, notice that with zero transaction costs then $e^T x = 1$ and we recover the standard risk measurement. So our choice does pass the first test required of any theoretical extension; recover the previous result. This choice also makes dimensional sense given the quadratic numerator.

Our choice of this fractional objective function also makes intuitive sense. For nonzero transaction costs, there are conflicting effects at work within the portfolio. For a given $\bar{x}$, the absolute amount of principal available for investment will decrease as the transaction cost percentage is increased. But in order to get the same payoff ($\mu^T x$) on a smaller amount of principal the investor will need to reach for higher returns. This should correlate to taking on higher levels of risk. Our fractional choice effectively boosts the risk measurement for these transaction cost depleted portfolios, and results in a portfolio with lower transaction costs.

Theorem 2 *If the optimal solution to* (8) *is not optimal to (5) then the transaction costs paid to implement the optimal solution to* (8) *will be less than those required by the optimal solution to (5).*

Proof: Let $\hat{x}$ solve (8) and let $\tilde{x}$ solve (5). The feasible regions to (8) and (5) are identical, so

$$\tilde{x}^T Q \tilde{x} < \hat{x}^T Q \hat{x} \quad \text{and} \quad \frac{\hat{x}^T Q \hat{x}}{(e^T \hat{x})^2} \leq \frac{\tilde{x}^T Q \tilde{x}}{(e^T \tilde{x})^2}.$$

It follows that $e^T \hat{x} > e^T \tilde{x}$. □

Problem (8) is equivalent to a convex optimization problem so it can be solved efficiently, as we show in §4. It is also shown in §4 that the optimal solution to (8) is frugal, even if there is no risk-free security, provided the return constraint is active at the optimal solution.

In addition to being frugal and having lower transaction costs than the optimal solution to (5), the optimal solution x^* to (8) possesses other attractive features. It is the portfolio of minimum risk among all portfolios achieving expected return E_0 and with transaction costs no greater than a certain level.

Theorem 3 *Let x^* be an optimal solution to* (8)*. Let $T = 1 - e^T x^*$ be the transaction costs of this portfolio. The portfolio x^* is the minimum risk portfolio among all portfolios with expected return at least E_0 and transaction costs no greater than T.*

Proof: For any portfolio $\tilde{x}$ feasible in (8), we have

$$\frac{{x^*}^T Q x^*}{(e^T x^*)^2} \leq \frac{\tilde{x}^T Q \tilde{x}}{(e^T \tilde{x})^2}$$

It follows that if $e^T \tilde{x} \geq e^T x^*$ then $\tilde{x}^T Q \tilde{x} \geq {x^*}^T Q x^*$. $\square$

The Markowitz mean-variance model produces an efficient frontier as the desired level of returns is varied. This frontier corresponds to the set of Pareto optimal solutions to the multiobjective portfolio optimization problem with the two competing objectives of minimizing the variance and maximizing the expected return. Transaction costs add an additional feature to the picture, and minimizing transaction costs can be regarded as a third competing objective. The set of Pareto optimal solutions is then a two-dimensional surface, which we call the Transaction Cost Efficient Frontier (TCEF).

Definition 2 *Let $g_i : I\!R^n \to I\!R$, $i = 1, \ldots, k$ be competing objective functions, all to be minimized. Let $g(x) := (g_1(x), \ldots, g_k(x))^T$. Let $S \subseteq I\!R^n$ be the set of feasible solutions. A feasible point $x^* \in S$ is a* Pareto optimal point *for the multiple objective problem* $\min\{g(x) : x \in S\}$ *if there is no $\tilde{x} \in S$ with $g(\tilde{x}) \leq g(x^*)$ and $g(\tilde{x}) \neq g(x^*)$.*

All three of our objective functions are convex. The feasible set of portfolios is

$$SP := \{x \in I\!R^n : e^T x + c(x - \bar{x}) \leq 1, Ax \leq 0\} \tag{9}$$

It follows that Pareto optimal points can be found by minimizing one objective while bounding the values of the other objectives. From Theorem 3, we have that the optimal solution to (8) is a Pareto optimal solution for any choice of E_0. As E_0 is varied, the set of such solutions traces out a curve on the surface of efficient solutions, which we call the Transaction Cost Efficient Frontier Curve (TCEFC). In Figure 1, we indicate the structure of the TCEFC for a nine-security problem, with three different choices for the transaction costs, namely zero costs, 3% costs for each buy and sell decision, and 5% costs for each buy and sell decision. The initial portfolio is equally weighted in the nine securities.

The optimal objective function is plotted against the value of E_0. Optimizing this quadratic program creates the situation where the $c = 0\%$ frontier extends furthest into the risk/return plane. Other transaction cost efficient frontier curves are pulled back from that limit as illustrated in Figure 1. It is apparent that transaction costs reduce the range of investment choice.

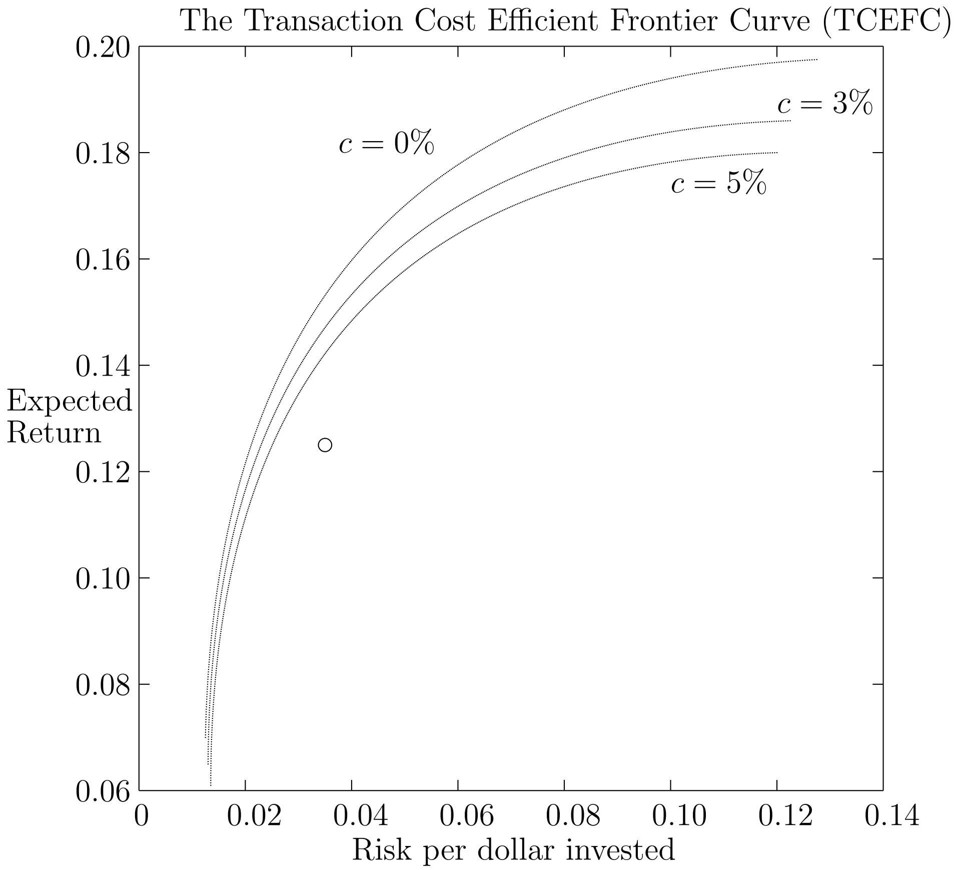

Figure 1: The initial portfolio is located by a circle. Notice that as the level of transaction costs c increases, the curves shift right. Increased transaction costs reduce investment choice.

The solution to (3) also sits on the efficient frontier, and traces out a curve as E_0 is varied. The optimal frugal solution to (5) is also a Pareto optimal solution.

The transaction costs are deterministic, based on the choice of portfolio. The returns and covariances are stochastic. These different elements should be handled differently, and this suggests that transaction costs should be emphasized to a greater extent than given in (5). Solving (8) is a mechanism for giving greater emphasis to the deterministic loss of transaction costs.

4 Solving the fractional quadratic program

The fractional objective $f(x)$ from (7) can be made quadratic using the technique of replacing the denominator by the square of the reciprocal of a variable. This is a straightforward extension of the technique of Charnes and Cooper [14] for fractional programs where the objective is a ratio of linear functions and the constraints are linear. Schaible [43] generalized the method of [14] to problems where the objective function is a ratio of convex functions, a setting which includes (7) as a special case. Let

$$t := \frac{1}{e^T x} \tag{10}$$

and then define

$$\hat{x} := tx. \tag{11}$$

Note that since $c(x)$ is a nonnegative function, we must have $t \geq 1$. The constraints of (8) can be multiplied through by t. We also need to include the constraint $e^T\hat{x} = 1$, which is equivalent to (10). Thus, the fractional quadratic program (8) is equivalent to the convex program with quadratic objective:

$$\begin{array}{llrrcl} \min_{\hat{x},t} & \frac{1}{2}\hat{x}^T Q\hat{x} & & & & \\ \text{s.t.} & \mu^T\hat{x} & & - \; E_0 t & \geq & 0 \\ & e^T\hat{x} & + \; \hat{c}(\hat{x},t) & - \; t & \leq & 0 \\ & e^T\hat{x} & & & = & 1 \\ & A\hat{x} & & & \leq & 0 \end{array} \tag{12}$$

where

$$\hat{c}(\hat{x},t) := tc(\frac{1}{t}\hat{x} - \bar{x}). \tag{13}$$

Once we find a solution $(\hat{x}^*, t^*)$ to (12), we can obtain a solution x^* to the original problem (8) by rescaling $\hat{x}^*$ so $x^* = \frac{1}{t^*}\hat{x}^*$.

We make the following Slater constraint qualification assumption, so that any local minimizer is a Karush-Kuhn-Tucker (KKT) point:

Assumption 2 *There exists a feasible solution $\hat{x}$, t for* (12) *with* $e^T\hat{x} + \hat{c}(\hat{x},t) - t < 0$.

In the next two theorems, we show that the optimal solution to (12) satisfies the transaction cost constraint at equality if the return constraint has a positive multiplier at optimality, and that even if the return constraint is inactive there is an optimal solution satisfying the transaction cost constraint at equality. The proofs of these theorems do not exploit the structure of the objective function, so the results hold even if $\frac{1}{2}\hat{x}^T Q\hat{x}$ is replaced by another measure $g(\hat{x})$ of risk.

Theorem 4 *If the optimal solution to* (12) *has a strictly positive Karush-Kuhn-Tucker multiplier* w *for the return constraint* $\mu^T\hat{x} - E_0 t \geq 0$ *then the optimal solution satisfies* $e^T\hat{x} + \hat{c}(\hat{x}, t) - t = 0$.

Proof: If $e^T\hat{x} + \hat{c}(\hat{x}, t) - t < 0$ then the t component of the KKT optimality conditions for (12) requires $E_0 w = 0$, that is, $w = 0$. □

If an investor just wishes to minimize risk, with little concern for expected return, then the value of E_0 can be set to a low number, with the result that the return constraint may be satisfied strictly in the solution to (12). Alternatively, it may be that the presence of the homogeneous constraints $Ax \leq 0$ results in an optimal solution that satisfies the return constraint strictly. If the return constraint is not active at the optimal solution, then an optimal solution may be wasteful. Nonetheless, there is an optimal solution that is frugal, and that solution can be found efficiently. If we modify t while fixing $\hat{x} = \hat{x}^*$ then the objective function value in (12) does not change. Reducing t will reduce the amount spent in transaction costs.

Theorem 5 *There exists a frugal optimal solution to* (12).

Proof: Let $\hat{x}^*$ and t^* be an optimal solution to (12) with $e^T\hat{x}^* + t^* c(\frac{1}{t^*}\hat{x}^* - \bar{x}) - t^* < 0$. If $\hat{x}^* \neq \bar{x}$ then $\hat{x} = \hat{x}^*$, $t = 1$ violates the transaction cost constraint. Let $t_{min} := \min\{t : t \geq 1, t < t^*, e^T\hat{x}^* + tc(\frac{1}{t}\hat{x}^* - \bar{x}) - t = 0\}$. This is well-defined since the transaction cost function c is continuous. Note that $\hat{x} = \hat{x}^*$, $t_{min} \leq t \leq t^*$ must be optimal in (12): it has value equal to the optimal value, it satisfies $e^T\hat{x} = 1$ and $Ax \leq 0$, and decreasing t from t^* increases the slack in the return constraint. □

Problem (12) is a convex optimization problem in the variables $\hat{x}$ and t. The subgradients of the transaction cost constraint can be obtained from the subgradients of c.

Theorem 6 *Problem* (12) *is a convex optimization problem.*

Proof: The transaction cost function $\hat{c}(\hat{x}, t)$ is convex if and only if it has a subgradient at each point $(\hat{x}, t)$. Given $\hat{x}$ and $t > 0$, let ξ denote a subgradient of $c(x)$ at $x = \frac{1}{t}\hat{x} - \bar{x}$. We show there exists a subgradient of $\hat{c}(\hat{x}, t)$. For any $s > 0$ and any z, we have:

$$\hat{c}(z, s) = sc(\frac{1}{s}z - \bar{x})$$

$$
\begin{aligned}
&\geq \quad sc(\frac{1}{t}\hat{x} - \bar{x}) + s\xi^T(\frac{1}{s}z - \frac{1}{t}\hat{x}) \\
&= \quad tc(\frac{1}{t}\hat{x} - \bar{x}) + c(\frac{1}{t}\hat{x} - \bar{x})(s-t) + \xi^T(z - \hat{x}) - \frac{\xi^T\hat{x}}{t}(s-t) \\
&= \quad \hat{c}(\hat{x}, t) + \begin{bmatrix} \xi \\ c(\frac{1}{t}\hat{x} - \bar{x}) - \frac{\xi^T\hat{x}}{t} \end{bmatrix}^T \begin{bmatrix} z - \hat{x} \\ s - t \end{bmatrix}
\end{aligned}
$$

Thus, $\hat{c}(\hat{x}, t)$ is convex for $t > 0$, and so (12) is a convex optimization problem. □

The efficient frontier for (5) can also be obtained by maximizing the return, subject to a threshold on allowable risk. This is the approach given in Lobo et al. [31], for example. Analogously, the efficient frontier arising from (8) can be found by solving problems of the form

$$
\begin{array}{lrcl}
\max & \mu^T x & & \\
\text{s.t.} & ||x^T Q x|| & \leq & 2Re^T x \\
 & e^T x + c(x - \bar{x}) & \leq & 1 \\
 & Ax & \leq & 0.
\end{array} \tag{14}
$$

for different values of allowable risk R, where $||x^T Qx||$ denotes the Euclidean norm of $x^T Qx$. This is a second-order cone program if c is piecewise linear, and so can be solved efficiently using an interior point method; see, for example, Andersen et al. [4].

Various different transaction cost functions are considered in the rest of this section. In particular, piecewise linear cost functions (including proportional transaction costs) are the subject of §4.1, quadratic and piecewise quadratic functions are discussed in §4.2, market impact costs are detailed in §4.3, and an approach to nonconvex transaction cost functions is proposed in §4.4. We also consider alternative measures of risk in §4.5.

4.1 Piecewise linear cost functions

If $c(x)$ is a piecewise linear function of x then $\hat{c}(\hat{x}, t)$ is also piecewise linear, as a function of $\hat{x}$ and t. In this case, the transaction cost function can be represented as a sum of convex functions, and each of those functions can be represented as the maximum of a set of linear functions, so

$$
c(x) = \sum_{k \in K} c_k(x) \tag{15}
$$

for some index set K, and

$$
c_k(x) = \max_{j \in J(k)} \{ {c_k^j}^T x + v_k^j \} \tag{16}
$$

for an index set $J(k)$ for each $k \in K$, and where c_k^j is an n-vector and v_k^j is a scalar. The cost functions of (2) and (4) can be placed in this framework. The cost function $\hat{c}(\hat{x}, t)$ then has this same structure. In particular,

$$
\hat{c}(\hat{x}, t) = tc(\frac{1}{t}\hat{x} - \bar{x}) = \sum_{k \in K} \hat{c}_k(\hat{x}, t) \tag{17}
$$

with

$$\hat{c}_k(x,t) = \max_{j\in J(k)} \{ {c_k^j}^T \hat{x} + (v_k^j - {c_k^j}^T \bar{x})t \}. \tag{18}$$

Problem (12) can then be written equivalently as

$$\begin{array}{llllllll}
\min_{\hat{x},\hat{u},\hat{v},t} & \frac{1}{2}\hat{x}^T Q\hat{x} & & & & & & \\
\text{s.t.} & \mu^T\hat{x} & & & - & E_0 t & \geq & 0 \\
& e^T\hat{x} & + & \sum_{k\in K} z_k & - & t & \leq & 0 \\
& {c_k^j}^T\hat{x} & - & z_k & + & (v_k^j - {c_k^j}^T\bar{x})t & \leq & 0 \quad \forall k \in K,\ \forall j \in J(k) \\
& e^T\hat{x} & & & & & = & 1 \\
& A\hat{x} & & & & & \leq & 0
\end{array} \qquad (PLP)$$

This is a quadratic programming problem, with a convex quadratic objective function and linear constraints, so it can be solved efficiently.

4.2 Quadratic and piecewise quadratic transaction costs

Convex quadratic terms in the transaction cost function $c(x)$ result in the corresponding portion of $\hat{c}(\hat{x},t)$ being a second order cone constraint. For a quadratic cost function $c(x) = x^T M x$ where M is a square symmetric positive semidefinite matrix, the transaction cost constraint in (12), namely

$$e^T\hat{x} + \hat{c}(\hat{x},t) - t \leq 0 \tag{19}$$

can be written equivalently as

$$t^2 - t \geq (\hat{x} - t\bar{x})^T M (\hat{x} - t\bar{x})$$

or equivalently

$$(t - 0.5)^2 \geq 0.5^2 + (L^T(\hat{x} - t\bar{x}))^T (L^T(\hat{x} - t\bar{x}))$$

since $e^T\hat{x} = 1$, and where the matrix L satisfies $LL^T = M$. This is a second order cone constraint and so can be handled efficiently.

Piecewise quadratic convex cost functions are also of interest. For example, the transaction cost for a security may take the form

$$c_i(x_i - \bar{x}_i) = \begin{cases} 0.1(x_i - \bar{x}_i)^2 & \text{if } 0 \leq x_i - \bar{x}_i \leq 0.2 \\ 0.004 + 0.3(x_i - \bar{x}_i - 0.2)^2 & \text{if } 0.2 \leq x_i - \bar{x}_i \\ 0.15(x_i - \bar{x}_i)^2 & \text{if } x_i \leq \bar{x}_i \end{cases}$$

representing the effect of scarcity of security i. By handling each of the three regions separately, this can be expressed equivalently as

$$c_i(x_i - \bar{x}_i) = \left\{ \begin{array}{lccccccc} \min & 0.1x_{i1}^2 & + & 0.3x_{i2}^2 & + & 0.15x_{i3}^2 & & \\ \text{subject to} & x_{i1} & + & x_{i2} & + & x_{i3} & = & x_i - \bar{x}_i \\ & 0 \leq x_{i1} \leq 0.2, & & 0 \leq x_{i2}, & & 0 \geq x_{i3} & & \end{array} \right\}$$

Such individual security transaction costs can be summed. It may be that the transaction cost function is proportional in certain regions. More general quadratic functions can also be allowed. A general piecewise quadratic convex transaction cost function, with each region being polyhedral, can be expressed as

$$c(x) = \left\{ \begin{array}{lrcll} \min & \sum_{j \in J} {x^j}^T M_j x^j + g_j^T x^j & & & \\ \text{subject to} & \sum_{j \in J} x^j & = & x & \\ & P_j x^j & \leq & u_j & \forall j \in J \end{array} \right\} \tag{20}$$

where J is an index set of the polyhedral regions, each variable x^j is an n-vector, each g_j and u_j is an appropriately dimensioned vector, each M_j is a positive semidefinite matrix, and P_j is an appropriately dimensioned matrix. Note that x is expressed as a sum of points drawn from the different regions. We require that $u_j \geq 0$ so that $x^j = 0$ satisfies the polyhedral restriction $P_j x^j \leq u_j$. There is no constant term in the objective since $c(0) = 0$ by Assumption 1.

The transaction cost constraint (19) can then be expressed equivalently as the system of constraints:

$$\begin{array}{rcrcrcl} e^T \hat{x} & + & t \sum_{j \in J} {x^j}^T M_j x^j + t g_j^T x^j & - & t & \leq & 0 \\ \frac{1}{t} \hat{x} & - & \sum_{j \in J} x^j & & & = & \bar{x} \\ & & P_j x^j & & & \leq & u_j \quad \forall j \in J \end{array}$$

Since $t > 0$,we can make the change of variables $\hat{x}^j = t x^j$ for $j \in J$. Using $\hat{z}$ to give the quadratic part of the cost function results in the formulation:

$$\begin{array}{rcrcrcrcl} e^T \hat{x} & + & \hat{z} & + & g_j^T x^j & - & t & \leq & 0 \\ \hat{x} & & & - & \sum_{j \in J} \hat{x}^j & - & t\bar{x} & = & 0 \\ & & t\hat{z} & - & \sum_{j \in J} {\hat{x}^j}^T M_j \hat{x}^j & & & \geq & 0 \\ & & & & P_j \hat{x}^j & - & t u_j & \leq & 0 \quad \forall j \in J \end{array}$$

This is a mixture of linear constraints and a rotated second order cone constraint, so again it can be handled efficiently.

4.3 Market impact costs

Market impact costs are typically modeled as

$$c(x - \bar{x}) = \sum_{i=1}^{n} c_i(x_i - \bar{x}_i) = \sum_{i=1}^{n} \kappa_i |x_i - \bar{x}_i|^{1+\gamma}$$

where $\kappa_i \geq 0$, $i = 1, \ldots, n$ are parameters for each security, and usually $\gamma = 0.5$.

The transaction cost constraint

$$e^T x + c(x - \bar{x}) \leq 1$$

from (5) is equivalent to the family of constraints

$$\begin{aligned} e^T x + \sum_{i=1}^{n} \kappa_i w_i &\leq 1 \\ |x_i - \bar{x}_i|^{1+\gamma} &\leq w_i \text{ for } i = 1, \ldots, n. \end{aligned} \tag{21}$$

This approach can be generalized to allow different parameters κ_i^+ and κ_i^- for the buy and sell decisions. The constraint (21) can be expressed as a family of second order constraints for rational choices of γ, using techniques discussed by Alizadeh and Goldfarb [2]. For example, with $\gamma = 0.5$, the constraint

$$(x_i - \bar{x}_i)^{1.5} \leq w_i, \quad x_i \geq \bar{x}_i$$

is equivalent to

$$(x_i - \bar{x}_i)^4 \leq w_i^2(x_i - \bar{x}_i), \quad x_i \geq \bar{x}_i$$

which in turn is equivalent to

$$(x_i - \bar{x}_i)^2 \leq uw_i, \quad u^2 \leq (x_i - \bar{x}_i), \quad x_i \geq \bar{x}_i, \quad u \geq 0.$$

The transaction cost constraint in (12) can also be reformulated as a family of second order cone constraints for positive rational γ. The individual component constraints take the form

$$t(\frac{1}{t}\hat{x}_i - \bar{x}_i)^{1+\gamma} \leq w_i, \quad \frac{1}{t}\hat{x}_i \geq \bar{x}_i$$

for a buy decision, which is equivalent to

$$(\hat{x}_i - t\bar{x}_i)^{1+\gamma} \leq w_i t^{\gamma}, \quad \hat{x}_i \geq t\bar{x}_i$$

since t is strictly positive. With $\gamma = 0.5$, we obtain an equivalent system:

$$(\hat{x}_i - t\bar{x}_i)^4 \leq w_i^2 t(\hat{x}_i - t\bar{x}_i), \quad \hat{x}_i \geq t\bar{x}_i,$$

which in turn is equivalent to

$$(\hat{x}_i - t\bar{x}_i)^2 \leq uw_i, \quad u^2 \leq t(\hat{x}_i - t\bar{x}_i), \qquad \hat{x}_i \geq t\bar{x}_i, \quad u \geq 0.$$

Similarly, with $\gamma = 0.6$, we obtain an equivalent system:

$$(\hat{x}_i - t\bar{x}_i)^8 \leq w_i^5 t^3, \quad \hat{x}_i \geq t\bar{x}_i,$$

which in turn is equivalent to

$$(\hat{x}_i - t\bar{x}_i)^2 \leq u_1 w_i, \quad u_1^2 \leq u_2 t, \quad u_2^2 \leq w_i t, \quad \hat{x}_i \geq t\bar{x}_i, \quad u_1, u_2 \geq 0.$$

4.4 Nonconvex transaction costs

In this subsection, we consider relaxing Assumption 1.1. Nonconvex transaction costs of interest include piecewise linear cost functions used when there are price breaks for large transactions.

If the transaction cost function is continuous and has convex level sets then problem (8) is still equivalent to (12) and may well still be a useful formulation. Further, it may be possible to solve it efficiently using a cutting plane method, with a polyhedral representation of the convex set of feasible solutions developed iteratively. In general, given a convex level set $C = \{y \in \mathbb{R}^m : g(y) \leq 0\}$ for some function $g(y)$, if $\tilde{y}$ satisfies $g(\tilde{y}) \geq 0$ and $g(y)$ is differentiable at $\tilde{y}$, then $C \subseteq \{y : \nabla g(\tilde{y})^T (y - \tilde{y}) \leq 0\}$.

In such an approach, the transaction cost constraint can initially be relaxed to $e^T \hat{x} \leq t$. If the optimal solution $(\hat{x}^*, t^*)$ to the relaxation violates the transaction cost constraint, and if $\hat{c}$ is differentiable at $(\hat{x}^*, t^*)$, the linear constraint

$$(e + \nabla_{\hat{x}} \hat{c}(\hat{x}^*, t^*))^T (\hat{x} - \hat{x}^*) + (\frac{\partial \hat{c}}{\partial t}(\hat{x}^*, t^*) - 1)(t - t^*) \leq 0$$

can be added to the formulation. A similar constraint can be found even at points where $\hat{c}(\hat{x}^*, t^*)$ is not differentiable. (8) can thus be solved using the ellipsoid algorithm [26] or an interior point cutting plane algorithm [23, 36].

4.5 Other measures of risk

Problem (8) uses the classical Markowitz measure of risk of $x^T Q x$. Other measures of risk have been proposed in the literature; see, for example, Gondzio and Grothey [24] or Rockafellar and Uryasev [42] or El Ghaoui et al. [21]. In many cases the objective function of (8) can be modified similarly, and the rescaling of §4 can still be used to obtain a more tractable problem. Robust optimization approaches to portfolio optimization include Kim and Boyd [28]; it would be of interest to extend our framework to these approaches.

For example, the semivariance can be used in a manner similar to [24]. The semivariance counts only scenarios where the return is less than the expected return. Given a finite number of scenarios with known end-of-period values $(\mu^i)^T x$ and with probabilities p_i, constraints of the form

$$(\mu^i)^T x - \mu^T x + s_i^+ - s_i^- = 0, \qquad s_i^+, s_i^- \geq 0 \tag{22}$$

can be introduced, and the semivariance is equal to

$$SV := \sum_i p_i (s_i^+)^2.$$

The objective term $x^T Q x$ of (8) can be replaced by SV, and the change of variables used to obtain (12) can still be used, giving a quadratic objective function, with the additional constraints (22) still linear after the transformation. Note that Theorems 4–6 still hold in this setting.

5 The Sharpe Ratio

One measure used to choose between portfolios on the efficient frontier is the Sharpe Ratio. The portfolio that optimizes the Sharpe Ratio can be found by solving a fractional quadratic programming problem, and so it can be found using the techniques of §4. This leads to a curve on the efficient surface.

The Sharpe Ratio is defined as the ratio of the differential return to its standard deviation [44, 45]. To be precise, the return of a portfolio is the proportional increase in the value of the portfolio from the beginning to the end of the period. The differential return is the difference between the return of the portfolio and the return obtained through the risk-free security. The Sharpe Ratio is widely used in the financial community. As Sharpe [45] states,

> "When choosing one from among a set of funds to provide representation in a particular market sector, it makes sense to favor the one with the greatest predicted Sharpe Ratio".

The expected differential return is $\zeta := \rho - \rho_f e$ where ρ_f is the return from the risk-free security. We can assume without loss of generality that $\zeta > 0$ and Q is positive definite. If there are no transaction costs, maximizing the Sharpe Ratio is equivalent to minimizing the ratio of x^TQx and $(\zeta^T x)^2$. This is independent of the scale of x, so the portfolio that maximizes the Sharpe ratio can be found by solving the quadratic programming problem

$$\begin{array}{lrcl} \min & 0.5x^TQx & & \\ \text{subject to} & \zeta^T x & = & 1 \\ & Ax & \leq & 0 \end{array} \tag{23}$$

where the constraints $Ax \leq 0$ capture additional restrictions on allowable portfolios. This problem can alternatively be derived using the technique of §4, applied to a problem with an objective of a ratio of x^TQx and $(\zeta^T x)^2$.

Some care is needed when optimizing the Sharpe Ratio. For example, Goetzmann et al. [22] show that it can be improved in some situations by discarding strategies with high returns, since this reduces the risk denominator. Thus, it is advisable to control the allowable portfolios through additional constraints of the form $Ax \leq 0$. Without these additional constraints, the optimal solution is proportional to $Q^{-1}\zeta$.

The problem (23) can be modified if there are transaction costs. The level of allowable transaction costs can be explicitly limited. One possible formulation is the following:

$$\begin{array}{llcccccc} \min_{\hat{x},t} & \frac{1}{2}\hat{x}^TQ\hat{x} & & & & & & \\ \text{s.t.} & e^T\hat{x} & + & \hat{c}(\hat{x},t) & - & t & \leq & 0 \\ & \zeta^T\hat{x} & & & & & = & 1 \\ & & & \hat{c}(\hat{x},t) & & & \leq & T \\ & A\hat{x} & & & & & \leq & 0 \end{array} \tag{24}$$

where $\hat{c}(\hat{x},t)$ is defined in (13) and T is a positive scalar. The solution $\hat{x}$ to (24) needs to be divided by t and it then gives a curve on the Transaction Cost Efficient Frontier (TCEF) as T is varied. The expected value at the end of the period of the portfolio that maximizes this transaction cost limited version of the Sharpe ratio is $1/t$. The total amount paid in transaction costs is no greater than T/t.

For the example of 2.1, assume the risk-free return ρ_f is 1%. The optimal Sharpe Ratio with no transaction costs is 0.4954, achieved by placing approximately 78.6% of wealth in Asset A and 21.4% in Asset B. The expected differential return is 39.4%. With proportional transaction costs of 2% and with T sufficiently large, the same Sharpe Ratio is obtained, with approximately 1.1% of the wealth consumed by transaction costs. The expected differential return drops to 38.9%. With $T = 0.01$ the optimal Sharpe Ratio drops to 0.4815 and the expected return drops to 29.8%. Just 0.3% of wealth is consumed in transaction costs and 57.3% and 42.4% of the wealth is invested in Assets A and B respectively.

6 Computational Results

We discuss two portfolios in this section, a nine-security one due to Markowitz [34], and a portfolio consisting of the thirty stocks in the Dow Jones Industrial Average. Solving the problem (12) for different values of E_0 will give a transaction cost efficient frontier curve (TCEFC).

We investigated the Markowitz 9-security portfolio with all transaction costs equal to 3% and with all transaction costs equal to 5%, with the initial portfolio equally divided among the securities in both cases. The structure of the results is indicated in Figure 1. For the return and risk data, and more details of the results, see Braun [10]. A general observation is that the portfolios along the TCEFC are not simply related to the portfolios along the no transaction cost efficient frontier. Sometimes, entirely new securities are involved. Sometimes, buy and sell decisions are reversed. The introduction of costs changes the portfolio rebalancing problem dramatically and the optimal solutions are also quite different.

We applied our solution strategy to the problem of rebalancing portfolios composed of the 30 stocks which currently make up the Dow Jones Industrials Average. All securities were involved initially, with proportions varying from 1% to 5%. The buying and selling costs varied from security to security, from 0% to 5%. For more details on these experiments, see Braun [10]. As with the earlier example, the optimal solution was altered by the transaction costs.

7 Conclusions

The results of this paper will allow the incorporation of transaction costs into portfolio optimization problems, in a manner that leads to intuitive and sensible allocations. The model calculates the risk of the resulting portfolio, weighted by the amount invested after paying transaction costs. The model can be formulated as a convex optimization problem of size comparable to the model with no transaction costs, and it can often be solved efficiently.

From the computational results, it appears that the effect of transaction costs is more marked for relatively high levels of desired expected return, since the portfolio manager is then forced to perform a lot of rebalancing because only a few assets can meet the desired return requirement.

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