Rebalancing an Investment Portfolio in the Presence of Transaction Costs

John E. Mitchell[1]
Department of Mathematical Sciences,
Rensselaer Polytechnic Institute,
Troy, NY 12180.
mitchj@rpi.edu
http://www.rpi.edu/~mitchj

Stephen Braun[2]
Warren & Selbert, Inc.
Santa Barbara, CA 93101
brauns2@alum.rpi.edu

November 28, 2002.
Revised December 6, 2002.

Abstract

The inclusion of transaction costs is an essential element of any realistic portfolio optimization. In this paper, we consider an extension of the standard portfolio problem in which transaction costs are incurred to rebalance an investment portfolio. The Markowitz framework of mean-variance efficiency is used with costs modelled as a percentage of the value transacted. Each security in the portfolio is represented by a pair of continuous decision variables corresponding to the amounts bought and sold. In order to properly represent the variance of the resulting portfolio, it is necessary to rescale by the funds available after paying the transaction costs. We show that the resulting fractional quadratic programming problem can be solved as a quadratic programming problem of size comparable to the model without transaction costs. Computational results for two empirical datasets are presented.

Keywords: Portfolio optimization, transaction costs, rebalancing, quadratic programming

[1]Research partially supported by NSF Grant number CCR–9901822.
[2]Research supported in part by the National Science Foundation's VIGRE Program (Grant DMS-9983646).

1 Introduction

Constructing a portfolio of investments is one of the most significant financial decisions facing individuals and institutions. A decision-making process must be developed which identifies the appropriate weight each investment should have within the portfolio. The portfolio must strike what the investor believes to be an acceptable balance between risk and reward. In addition, the costs incurred when setting up a new portfolio or rebalancing an existing portfolio must be included in any realistic analysis. In this paper, we show that proportional transaction costs can be incorporated in a manner which makes the resulting optimization problem a quadratic program.

Essentially the standard portfolio optimization problem is to identify the optimal allocation of limited resources among a limited set of investments. Optimality is measured using a tradeoff between perceived risk and expected return. Expected future returns are based on historical data. Risk is measured by the variance of those historical returns.

When more then one investment is involved, the covariance among individual investments becomes important. In fact, any deviation from perfect positive correlation allows a beneficial diversified portfolio to be constructed. Efficient portfolios are allocations that achieve the highest possible return for a given level of risk. Alternatively, efficient portfolios can be said to minimize the risk for a given level of return. These ideas earned their inventor a Nobel Prize and have gained such wide acceptance that countless references could be cited; however, the original source is Markowitz [20], discussed in detail in his classic text [21]. A recent summary of the motivation for this model appears in Markowitz [22].

One standard formulation of the portfolio problem minimizes a quadratic risk measurement with a set of linear constraints specifying the minimum expected portfolio return, E_0, and enforcing full investment of funds. The decision variables x_i are the proportional weights of the i^{th} security in the portfolio. Here n securities are under consideration. Additionally, μ is the column vector of expected returns and Q is the positive semidefinite covariance matrix. Short selling is not allowed, so the proportions x_i are restricted to be nonnegative. This formulation is:

$$\begin{array}{lll} \min & \frac{1}{2}x^T Q x & \\ \text{s.t.} & \mu^T x & \geq E_0 \\ & e^T x & = 1 \\ & x & \geq 0, \end{array} \tag{1}$$

where e denotes the vector of all ones.

By varying the parameter E_0 and solving multiple instances of this problem, the set of efficient portfolios can be generated. This set, visualized in a risk/return plot, is called the efficient frontier. An investor may decide where along the efficient frontier (s)he finds an acceptable balance between risk and reward.

In this paper, we describe a method for finding an optimal portfolio when proportional transaction costs have to be paid. These costs vary linearly with the amount of a security bought or sold. Our method requires the solution of a quadratic program of similar size to the Markowitz model, and then the solution of a linear programming problem. Our method allows different costs for different securities, and different costs for buying and selling. Our model captures the feature that transaction costs are paid when a security is bought or sold and the transaction cost reduces the amount of that particular security that is available. In particular, both the risk and the return in our model are measured using the portfolio arising after paying the transaction costs.

The portfolio rebalancing problem has similarities to the index tracking problem [1, 8, 12]. See Zenios [29] for a discussion of portfolio optimization models. See Bertsimas et al. [5] for a case study in portfolio optimization. The optimal solution to the portfolio optimization problem is sensitive to the data Q and μ, so estimating this data accurately is the subject of current research; see Chopra and Ziemba [10] or Bengtsson and Holst [3] for example. Stochastic programming approaches to portfolio optimization have been investigated in [4, 11, 14, 15, 16, 24, 26, 27] and elsewhere; such approaches work with sets of scenarios.

This paper is organized as follows. We turn to the portfolio rebalancing problem in §2. First, we motivate the cost model and provide examples of costs that fit this model before introducing an essential change of variables. Alternative approaches to this problem are also discussed in §2. Our model is a quadratic programming problem, and the nature of the solutions to that problem is discussed in §3. If the return requirement is low, the nature of the solutions can change; this is the subject of §4. Computational results for two empirical datasets are presented in §5. Finally, we offer concluding remarks in §6.

2 Portfolio Rebalancing Problem

What we consider is an extension of the basic portfolio optimization problem in which transaction costs are incurred to rebalance a portfolio. That is, transactions are made to change an already existing portfolio, $\bar{x}$, into a new and efficient portfolio, x. A portfolio may need to be rebalanced periodically simply as updated risk and return information is generated with the passage of time. Further, any alteration to the set of investment choices would necessitate a rebalancing decision of this type.

In addition to the obvious cost of brokerage fees/commissions, here are two examples of other transaction costs that can be modeled in this way:

1. Capital gains taxes are a security-specific selling cost that can be a major consideration for the rebalancing of a portfolio.

2. Another possibility would be to incorporate an investor's confidence in the risk/return forecast as a subjective "cost". Placing high buying and selling costs on a security would favor maintaining the current allocation $\bar{x}$. Placing a

high selling cost and low buying cost could be used to express optimism that a security may outperform its forecast.

Let u_i and v_i represent the amount bought and sold (respectively) of security i. The amount invested in each of the securities will be

$$x = \bar{x} + u - v. \tag{2}$$

We assume proportional transaction costs. Let c_{B_i} and c_{S_i} denote the transaction cost of buying and selling one unit of security i, respectively. We assume $0 \leq c_B < e$, $0 \leq c_S < e$ and $c_S + c_B > 0$. We let x_0 denote the total amount spent on transaction costs, so

$$x_0 = c_B^T u + c_S^T v. \tag{3}$$

The total amount invested in the securities, after paying transaction costs, will be $1 - x_0$. We obtain the constraint

$$e^T x = 1 - c_B^T u - c_S^T v.$$

Exploiting the fact that $e^T \bar{x} = 1$, equation (2) immediately gives

$$(c_B + e)^T u + (c_S - e)^T v = 0. \tag{4}$$

This equation can be used in place of (1) to give a model for minimizing the variance of the resulting portfolio subject to meeting an expected return of $E_0 > 0$ in the presence of proportional transaction costs. The resulting model is

$$\begin{aligned}
\min \quad & \tfrac{1}{2} x^T Q x & & \quad (5)\\
\text{s.t.} \quad & \mu^T x & \geq E_0 & \quad (6)\\
& x - u + v & = \bar{x} & \quad (7)\\
& (c_B + e)^T u + (c_S - e)^T v & = 0 & \quad (8)\\
& u, v, x \geq 0. & & \quad (9)
\end{aligned}$$

A user might also require restrictions such as limiting the proportion of assets that can be invested in a group of securities. We can express this as a homogeneous constraint on x. For example, if security 1 must constitute no more than 10% of the resulting portfolio, we can impose the constraint

$$9x_1 - \sum_{i=2}^{n} x_i \leq 0.$$

We allow homogeneous constraints of the form $Ax \leq 0$ in our model, where A is an $m \times n$ matrix.

To this point, we have been optimizing the standard risk measure for efficient frontiers, that is:

$$\frac{1}{2} x^T Q x.$$

When there are no transaction costs to be paid, one dollar is always available for investment, i.e. ($\sum_{i=1}^{n} x_i = 1$). This assumption is implicit in the standard risk measure. However, for nonzero transaction costs that implicit assumption is no longer valid. One dollar is not available for investment, costs will be paid to rebalance. The appropriate objective is therefore

$$f(x) := \frac{\frac{1}{2} x^T Q x}{(1 - x_0)^2}. \tag{10}$$

Here x_0 is again the amount paid in transaction costs. Therefore $(1 - x_0)$ is the actual amount available for investment, so we are choosing to scale the standard risk measurement by the square of the dollar amount actually invested.

This gives the fractional quadratic programming problem (FQP) which we will solve to find the optimal portfolio for a given expected return.

$$\begin{array}{lccccccc}
\min & \frac{x^T Q x}{2(1 - c_B^T u - c_S^T v)^2} & & & & & & \\
\text{s.t.} & \mu^T x & & & & & \geq & E_0 \\
(FQP) & x & - & u & + & v & = & \bar{x} \\
 & & & (c_B + e)^T u & + & (c_S - e)^T v & = & 0 \\
 & Ax & & & & & \leq & 0 \\
 & & & & & u, v, x & \geq & 0.
\end{array}$$

Analytically, notice that with zero transaction costs then $x_0 \equiv 0$ and we recover the standard risk measurement. So our choice does pass the first test required of any theoretical extension; recover the previous result. This choice also makes dimensional sense given the quadratic numerator.

Our choice of this fractional objective function also makes intuitive sense. For nonzero transaction costs, there are conflicting effects at work within the portfolio. Thinking of fixed $\bar{x}$ then as the transaction cost percentage is increased, you expect smaller absolute amounts of principal will be available for investment. But in order to get the same payoff ($\mu^T x$) on a smaller amount of principal you will need to reach for higher returns. This should correlate to taking on higher levels of risk. Our fractional choice effectively boosts the risk measurement for these transaction cost depleted portfolios.

Without the denominator in the objective function, it is possible to obtain optimal solutions which involve both buying and selling a particular security, so both $u_i > 0$ and $v_i > 0$ for some security i. Of course, in practice this is not a desirable strategy. Nonetheless, it does reduce the measure of risk $\frac{1}{2} x^T Q x$. The incidence of solutions of this form is particularly noticeable for low values of E_0. Normalizing by the amount invested in securities prevents this undesirable outcome, as we shall see.

The fractional objective $f(x)$ can be made quadratic using the technique of replacing the denominator by the square of the reciprocal of a variable. This is a straightforward extension of the technique of Charnes and Cooper [9] for fractional programs where the objective is a ratio of linear functions and the constraints are linear. Let

$$t := \frac{1}{1 - c_B^T u - c_S^T v} \tag{11}$$

and then define

$$\hat{u} := tu, \quad \hat{v} := tv, \quad \hat{x} := tx. \tag{12}$$

Note that since u and v are constrained to be nonnegative, we must have $t \geq 1$. Note that we now have $t - c_B^T \hat{u} - c_S^T \hat{v} = 1$. The constraints (6)–(8) can be multiplied through by t. Thus, the fractional quadratic program (FQP) is equivalent to the quadratic programming problem (QP):

$$\begin{array}{llcrcrcrcl}
\min_{\hat{x},\hat{u},\hat{v},t} & \frac{1}{2}\hat{x}^T Q \hat{x} & & & & & & & & \\
\text{s.t.} & \mu^T \hat{x} & & & & & - & E_0 t & \geq & 0 \\
(QP) & \hat{x} & - & \hat{u} & + & \hat{v} & - & \bar{x} t & = & 0 \\
 & & & (c_B + e)^T \hat{u} & + & (c_S - e)^T \hat{v} & & & = & 0 \\
 & & - & c_B^T \hat{u} & - & c_S^T \hat{v} & + & t & = & 1 \\
 & A\hat{x} & & & & & & & \leq & 0 \\
 & & & & & & & \hat{x}, \hat{u}, \hat{v}, t & \geq & 0.
\end{array}$$

Once we find a solution $(\hat{x}^*, \hat{u}^*, \hat{v}^*, t^*)$ to (QP), we can obtain a solution (x^*, u^*, v^*) to the original problem (FQP) by rescaling $\hat{x}$, $\hat{u}$ and $\hat{v}$, so $x^* = \frac{1}{t^*}\hat{x}^*$, $u^* = \frac{1}{t^*}\hat{u}^*$, and $v^* = \frac{1}{t^*}\hat{v}^*$.

The efficient frontier is found by optimizing (QP) for different values of E_0. In Figure 1, we graph the efficient frontier for a nine-security problem, with three different choices for the transaction costs, namely zero costs, 3% costs for each buy and sell decision, and 5% costs for each buy and sell decision. The initial portfolio is equally weighted in the nine securities. The optimal objective function is plotted against the value of E_0. Optimizing this quadratic program creates the situation where the $c = 0\%$ frontier extends furthest into the risk/return plane. Other transaction cost efficient frontiers, abbreviated TCEF, are pulled back from that limit as seen in Figure 1. It is apparent that transaction costs reduce the range of investment choice.

The model (QP) is the one that we propose to solve in order to find an optimal level of risk for a given level of return in the presence of proportional transaction costs. In the next two sections we examine properties of the solutions to (QP). First, we conclude this section by considering other methods for portfolio rebalancing that have been discussed in the literature.

Adcock and Meade [1] add a linear term for the costs to the original Markowitz quadratic risk term and minimize this quantity. This requires finding an appropriate balance between the transaction costs and the risk. The model assumes a fixed rate of transaction costs across securities. The risk is measured in terms of the adjusted portfolio before transaction costs are paid.

Konno and Wijayanayasake [18] consider a cost structure that is considerably more involved than ours, with the result that the model is far harder to solve.

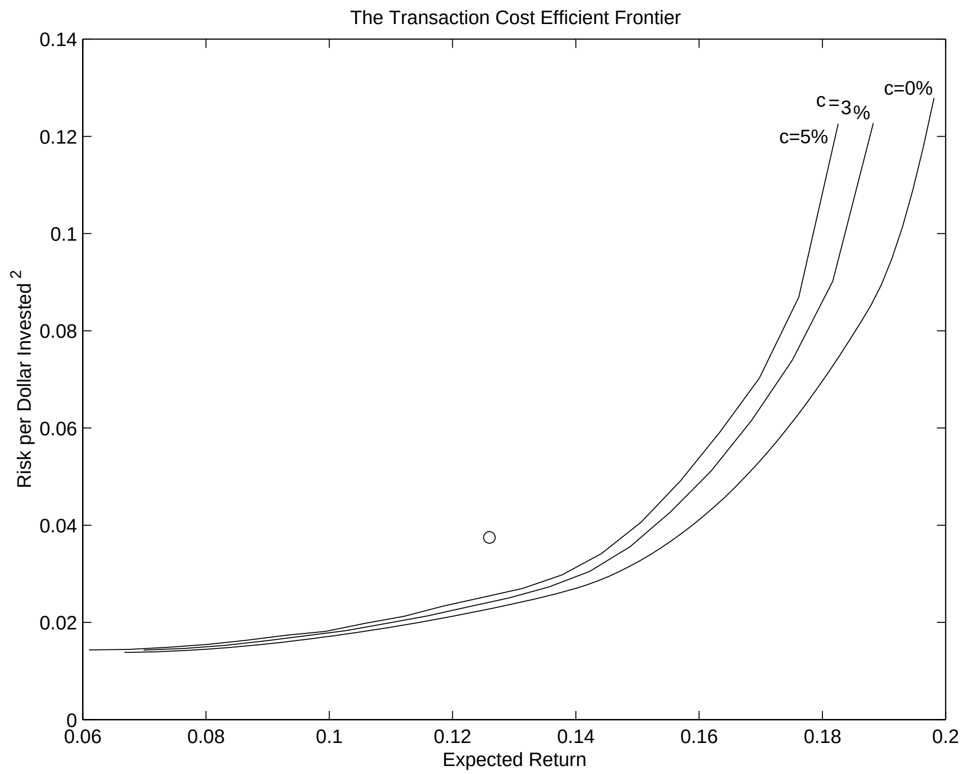

Figure 1: The initial portfolio is located by a circle. Notice that as the level of transaction costs c increases, the curves shift left. Increased transaction costs reduce investment choice.

Yoshimoto [28] considers a similar transaction cost model to ours and proposes a nonlinear programming algorithm to solve the problem. Computational results indicate that ignoring transaction costs can result in inefficient portfolios.

An alternative model is to reduce the vector of expected returns μ by the transaction costs. This gives a model of the form

$$\begin{array}{llcrcrcl} \min & \frac{1}{2}x^TQx & & & & & & \\ \text{s.t.} & \mu^T\bar{x} & + & (\mu - c_B)^T u & - & (\mu + c_S)^T v & \geq & E_0 \\ & x & - & u & + & v & = & \bar{x} \\ & e^T x & & & & & = & 1 \\ & Ax & & & & & \leq & 0 \\ & & & & & u, v, x & \geq & 0. \end{array}$$

The method implicitly assumes that transaction costs are paid at the end of the period, impacting both the risk and the return. If the transaction costs must be paid at the beginning of the period then care must be taken in the sale of assets to pay the transaction costs, in order to ensure that the resulting portfolio has securities in the same proportion. Further, the return calculation assumes a return on the amount paid in transaction costs, so this constraint needs to be modified.

If the only transaction cost is a fixed cost per transaction then transaction costs can be controlled by placing an upper bound on the number of transactions. This gives rise to a quadratically constrained integer programming problem. This approach has been investigated widely; see, for example, Perold [25], Bienstock [6] or Lee and Mitchell [19]. The presence of the integrality restriction makes the formulation far harder to solve than the one presented in this paper. Placing an explicit limit on the number of transactions is only an indirect way to limit transaction costs in the case of proportional transaction costs and may well result in suboptimal solutions. The constraint is rather crude. For example, it would require modification in the case that transaction costs differ from one security to another.

There has been some interest in portfolios that can be modified continuously, starting with Merton [23], with subsequent research including that by Albeverio et al [2], Davis and Norman [13], and Kamin [17]. These methods can only solve problems with a very limited number of securities.

3 Optimal solutions to (QP)

We have introduced variables to both buy u and sell v each security. We have not imposed an explicit constraint requiring that if a certain security is bought then it cannot also be sold. Both buying and selling a security would not be a desirable strategy in practice, but it might decrease the risk measure $\hat{x}^TQ\hat{x}$. We call a solution $(\hat{u}, \hat{v})$ *complementary* if it satisfies $\hat{u}^T\hat{v} = 0$, that is, if no stock is both bought and sold. In this section, we show that if the return constraint $\mu^T\hat{x} - E_0 t \geq 0$ is active at the optimal solution to (QP) then the optimal solution must be complementary.

If the return constraint is not active at the optimal solution, then it is possible that an optimal solution will not be complementary. We discuss this situation in §4, where we also show that a complementary solution can always be found efficiently even in this situation.

The Karush-Kuhn-Tucker (KKT) optimality conditions for the quadratic program (QP) require that $(\hat{x}, \hat{u}, \hat{v}, t)$ be primal feasible and that the following equations can be satisfied:

$$A^T\pi + Q\hat{x} - w\mu + s^r - s^x = 0 \quad (13)$$
$$-s^r + y(c_B + e) - zc_B - s^u = 0 \quad (14)$$
$$s^r + y(c_S - e) - zc_S - s^v = 0 \quad (15)$$
$$-wE_0 - \bar{x}^T s^r + z = 0 \quad (16)$$
$$w(\mu^T\hat{x} - E_0 t) = 0 \quad (17)$$
$$\hat{x}^T s^x = 0 \quad (18)$$
$$\hat{u}^T s^u = 0 \quad (19)$$
$$\hat{v}^T s^v = 0. \quad (20)$$

Here, w is a nonnegative scalar, y and z are free scalars, π is a nonnegative m-vector, s^r is a free n-vector, and s^x, s^u, and s^v are nonnegative n-vectors. We have exploited the fact that $t \geq 1$ in any feasible solution in deriving (16).

Assume that $\hat{u}_i > 0$ and $\hat{v}_i > 0$ for some security i. We argue that we must then have $w = 0$ for the KKT conditions to hold. If $\hat{u}_i > 0$ and $\hat{v}_i > 0$, it follows from (19) and (20) that $s^u_i = s^v_i = 0$. Adding together the ith components of (14) and (15) gives

$$(y - z)(c_{Bi} + c_{Si}) = 0.$$

Since $c_S + c_B > 0$, we immediately obtain $y = z$. Now adding together the whole of (14) and (15) gives $s^u + s^v = 0$, so $s^u = s^v = 0$ and $s^r = ye$. Substituting into (16) for s^r and using the facts that $y = z$ and $e^T\bar{x} = 1$ gives $wE_0 = 0$. Thus, we have proved the following theorem.

Theorem 1 *If the optimal solution to (QP) has a strictly positive Karush-Kuhn-Tucker multiplier for the return constraint $\mu^T\hat{x} - E_0 t \geq 0$ then the optimal solution is complementary.*

This theorem does not make any assumptions about the rank of Q. In particular, the result still holds even if there is a risk-free security, that is, a security for which there is no variability in the return. Any vector in the nullspace of Q can be regarded as a risk-free security. By a change of variables, this can be regarded as a single security, and the corresponding row and column of Q for this security are zero.

4 Low return requirements

If an investor just wishes to minimize risk, with little concern for expected return, then the value of E_0 can be set to a low number, with the result that the return constraint may be satisfied strictly in the solution to (QP). Alternatively, it may be that the presence of the homogeneous constraints $Ax \leq 0$ results in an optimal solution that satisfies the return constraint strictly. If the return constraint is not active at the optimal solution, then an optimal solution may not be complementary. Nonetheless, there is an optimal solution that is complementary, and that solution can be found efficiently. We first give an example of such a situation in §4.1, we then show that alternative optimal solutions exist, including one that is complementary in §4.2, and then we show how the complementary solution can be found efficiently in §4.3.

4.1 An example

Consider, for example, a problem with two securities, with initial portfolio $\bar{x} = (0.5, 0.5)$, with buying and selling transaction costs of 1%, with desired expected return for the portfolio of $E_0 = 29$, and with expected returns for the securities of $\mu = (25, 35)$. Assume the covariance matrix is equal to

$$Q = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}.$$

Assume there are no side constraints of the form $Ax \leq 0$. We also impose the upper bounds on u and v of $e - \bar{x}$ and $\bar{x}$, respectively. The problem is then solved using CPLEX called from AMPL. The optimal solution value is reported as 0.7140204261, achieved by

$$u = \begin{bmatrix} 0.42077 \\ 0.48194 \end{bmatrix}, \quad v = \begin{bmatrix} 0.50000 \\ 0.42096 \end{bmatrix}$$

If we require that a security be either bought or sold, and not both, then the optimal solution value is reported as 0.7142653036, achieved by

$$u = \begin{bmatrix} 0 \\ 0.07063 \end{bmatrix}, \quad v = \begin{bmatrix} 0.07206 \\ 0 \end{bmatrix}.$$

If this problem is solved to higher precision, it has multiple optimal solutions with value 0.714286, including noncomplementary solutions and a complementary solution with $u_2 = 0.07061$ and $v_1 = 0.07204$. The complementary solution can be recovered from a noncomplementary solution, as we argue later.

4.2 Alternative optimal solutions

Let $w^* := (\hat{x}^*, \hat{u}^*, \hat{v}^*, t^*)$ be an optimal solution to (QP). Assume this point is a noncomplementary solution. In this section, we show how an alternative optimal solution that is complementary can be obtained efficiently. If we modify $\hat{u}$, $\hat{v}$, and t so that $\hat{x} = t\bar{x} + \hat{u} - \hat{v}$ does not change and t does not increase then the objective function value does not change and the expected return constraint and the constraint $A\hat{x} \leq 0$ are still satisfied. Therefore, we will look for modifications $\Delta\hat{u}$, $\Delta\hat{v}$, and Δt

to $\hat{u}$, $\hat{v}$, and t, respectively, that maintain feasibility and which leave $\hat{x}$ unchanged. The new point will be

$$\begin{aligned} \hat{u}' &:= \hat{u}^* + \alpha\Delta\hat{u} && (21)\\ \hat{v}' &:= \hat{v}^* + \alpha\Delta\hat{v} && (22)\\ t' &:= t^* + \alpha\Delta t && (23)\\ \hat{x}' &:= \hat{x}^* + \alpha(\Delta t\bar{x} + \Delta\hat{u} - \Delta\hat{v}) && (24) \end{aligned}$$

for some scalar α.

Theorem 2 *Let $\Delta\hat{u}$ and $\Delta\hat{v}$ satisfy $\Delta\hat{u} - \Delta\hat{v} = \bar{x}$ and $(c_B + c_S)^T\Delta\hat{u} = (c_S - e)^T\bar{x}$. Let $\Delta t = -1$. Let α be a scalar chosen so that $\hat{u}' \geq 0$ and $\hat{v}' \geq 0$. The point defined in equations (21)–(24) is optimal for (QP).*

Proof: We show first that the point is feasible in (QP), and then we show that $\hat{x}' = \hat{x}^*$.

We have

$$\begin{aligned} (c_B + e)^T\hat{u}' + (c_S - e)^T\hat{v}' &= \alpha((c_B + e)^T\Delta\hat{u} + (c_S - e)^T\Delta\hat{v}) \\ &\quad\text{from feasibility of } w^* \\ &= \alpha((c_B + e)^T\Delta\hat{u} + (c_S - e)^T(\Delta\hat{u} - \bar{x})) \\ &\quad\text{from an assumption of the theorem} \\ &= \alpha((c_B + c_S)^T\Delta\hat{u} - (c_S - e)^T\bar{x}) \\ &= 0 \quad\text{from an assumption of the theorem} \end{aligned}$$

and

$$\begin{aligned} -c_B^T\hat{u}' - c_S^T\hat{v}' + t' &= \alpha(-c_B^T\Delta\hat{u} - c_S^T\Delta\hat{v} + \Delta t) \\ &\quad\text{from feasibility of } w^* \\ &= \alpha(e^T(\Delta\hat{u} - \Delta\hat{v}) - 1) \\ &\quad\text{from the previous string of equalities and} \\ &\quad\text{from an assumption of the theorem} \\ &= \alpha(e^T\bar{x} - 1) \\ &\quad\text{from an assumption of the theorem} \\ &= 0 \quad\text{since } e^T\bar{x} = 1. \end{aligned}$$

It follows immediately from the assumptions $\Delta t = -1$ and $\Delta\hat{u} - \Delta\hat{v} = \bar{x}$ that $\hat{x}' = \hat{x}^*$, so the given point is optimal in (QP). □

If Q has full rank then any alternative optimal solution must satisfy $\hat{u} - \hat{v} + t\bar{x} = \hat{x}^*$. We can then show that any optimal solution to (QP) must take the form given in Theorem 2.

Theorem 3 *If Q has full rank then any optimal solution to (QP) must take the form given in (21)–(24) and the directions must satisfy the hypotheses of Theorem 2, after rescaling so that $\Delta t = -1$.*

Proof: Since Q has full rank, we must have $\Delta\hat{u} - \Delta\hat{v} + t\bar{x} = 0$.

If $\Delta t = 0$ then $\Delta\hat{u} = \Delta\hat{v}$, but this is impossible if $\Delta\hat{u}$ and $\Delta\hat{v}$ also satisfy $(c_B + e)^T\Delta\hat{u} + (c_S - e)^T\Delta\hat{v} = 0$, since $c_S + c_B > 0$.

If $\Delta t \neq 0$, we can assume $\Delta t = -1$, rescaling if necessary. It follows that $\Delta\hat{u} - \Delta\hat{v} = \bar{x}$. We also have that $(c_B + e)^T\Delta\hat{u} + (c_S - e)^T\Delta\hat{v} = 0$, so $(c_B + c_S)^T\Delta\hat{u} = (c_S - e)^T\bar{x}$. □

4.3 Finding a complementary solution

In this section, we show how any noncomplementary optimal solution can be converted into a complementary one by solving a linear programming problem. Assume we have a noncomplementary solution with $\hat{u}_i^* > 0$ and $\hat{v}_i^* > 0$. Define a direction as follows:

For $j \neq i$:

$$\Delta\hat{u}_j = \begin{cases} \bar{x}_j & \text{if } \hat{v}_j^* = 0 \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

$$\Delta\hat{v}_j = \begin{cases} -\bar{x}_j & \text{if } \hat{v}_j^* > 0 \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

For i :

$$\Delta\hat{u}_i = \frac{1}{c_{B_i} + c_{S_i}}((c_S - e)^T\bar{x} - \sum_{j\neq i}(c_{B_j} + c_{S_j})\Delta\hat{u}_j) \tag{27}$$

$$\Delta\hat{v}_i = \Delta\hat{u}_i - \bar{x}_i \tag{28}$$

For t :

$$\Delta t = -1. \tag{29}$$

We discuss properties of this direction in the next three lemmas.

Lemma 1 *If either $\hat{u}_j^*$ or $\hat{v}_j^*$ is zero then the given direction ensures that $\hat{u}_j'\hat{v}_j' = 0$ for any choice of α, where $\hat{u}'$ and $\hat{v}'$ are defined in (21) and (22), respectively.*

Proof: If $\hat{v}_j^* > 0$ then both $\hat{u}_j^*$ and $\Delta\hat{u}_j$ are zero, so $\hat{u}_j' = 0$. If $\hat{v}_j^* = 0$ then $\Delta\hat{v}_j = 0$ so $\hat{v}_j' = 0$. The result follows. □

Lemma 2 *For the given direction, both $\Delta\hat{u}_i$ and $\Delta\hat{v}_i$ are negative.*

Proof: We have $\Delta\hat{u}_j \geq 0$ for $j \neq i$. It follows that $\Delta\hat{u}_i < 0$ and therefore that $\Delta\hat{v}_i < 0$. □

Lemma 3 *The given direction satisfies the conditions of Theorem 2.*

Proof: For every component, we have $\Delta\hat{u} - \Delta\hat{v} = \bar{x}$. Further, from the definition of $\Delta\hat{u}_i$, we have $(c_B + c_S)^T \Delta\hat{u} = (c_S - e)^T \bar{x}$. □

It follows from these lemmas that the direction defined by (25)–(29) maintains feasibility and optimality, so the point given in (21)–(24) is optimal for any α for which it is nonnegative. Further, the direction does not introduce noncomplementarity in any component. The maximum positive possible step length α can be found using a minimum ratio test. If the first component to be driven to zero is either $\hat{u}'_i$ or $\hat{v}'_i$ then we have removed the noncomplementarity in the ith component, without introducing noncomplementarity in any other variable. It may be that some other component $\hat{v}'_j$ is driven to zero first. In this case, we calculate a new direction using (25)–(29) and repeat. If $\bar{x}_j > 0$ then, in the new direction, we have $\Delta\hat{u}_j > 0$ and $\Delta\hat{v}_j = 0$ which makes $\Delta\hat{u}_i$ and $\Delta\hat{v}_i$ more negative. Further, the number of negative components $\Delta\hat{v}_j$ is decreased by at least one. Thus, we immediately have the following theorem.

Theorem 4 *If $w^* = (\hat{x}^*, \hat{u}^*, \hat{v}^*, t)$ is an optimal solution to (QP) with k components having both $\hat{u}^*_i > 0$ and $\hat{v}^*_i > 0$ then a complementary solution can be found in $O(nk)$ steps. Each step requires the calculation of the direction given in closed form in (25)–(29) and the calculation of a minimum ratio. In particular, any optimal solution w^* can be converted into a complementary optimal solution in $O(n^2)$ steps.*

Recall that x_0 is the total amount spent on transaction costs. It follows from (11) that $t = 1/(1 - x_0)$ so $x_0 = 1 - (1/t)$, so reducing t will reduce the amount spent in transaction costs. If an optimal solution w^* to (QP) is noncomplementary, then the value t can be reduced using the direction given in (25)–(29). Therefore, minimizing t while leaving $\hat{x}$ unchanged will result in a complementary optimal solution. This suggests solving the following linear programming problem.

$$
\begin{array}{llcllcll}
\min & & & & & \Delta t & & \\
\text{s.t.} & -c_B^T \Delta\hat{u} & - & c_S^T \Delta\hat{v} & + & \Delta t & = & 0 \\
(LP^t) & \Delta\hat{u} & - & \Delta\hat{v} & + & \bar{x}\Delta t & = & 0 \\
 & \Delta\hat{u} & & & & & \geq & -\hat{u}^* \\
 & & & \Delta\hat{v} & & & \geq & -\hat{v}^*
\end{array}
$$

Note that a feasible solution to this linear program will also satisfy $(c_B + e)^T \Delta\hat{u} + (c_S - e)^T \Delta\hat{v} = 0$, since $e^T \bar{x} = 1$.

The procedure given above for finding a complementary optimal solution to (QP) is not affected by the presence of a risk-free security. Of course, the optimal solution is to place all assets in the risk-free security giving an objective function value of zero, if this is feasible. If the risk-free security is a combination of the original securities, then it may not be possible to achieve a risk-free portfolio, even with very low values for E_0. Theorem 3 assumes there are no risk-free securities, so alternative optimal solutions may exist that do not give the same value for $\hat{x}$. In this situation, a direction

$\Delta\hat{x}$ in the nullspace of Q may lead to alternative optimal solutions. The minimum transaction cost solution can be found by solving the following generalization of (LP^t):

$$
\begin{array}{llcccccccl}
\min & & & & & & & \Delta t & & \\
\text{s.t.} & Q\Delta\hat{x} & & & & & & & = & 0 \\
 & \Delta\hat{x} & - & \Delta\hat{u} & + & \Delta\hat{v} & - & \bar{x}\Delta t & = & 0 \\
(LP^{rf}) & & & (c_B+e)^T\Delta\hat{u} & + & (c_S-e)^T\Delta\hat{v} & & & = & 0 \\
 & & & -c_B^T\Delta\hat{u} & - & c_S^T\Delta\hat{v} & + & \Delta t & = & 0 \\
 & \mu^T\Delta\hat{x} & & & & & - & E_0\Delta t & \geq & E_0 t^* - \mu^T\hat{x}^* \\
 & A\Delta\hat{x} & & & & & & & \leq & -A\hat{x}^* \\
 & & & \Delta\hat{u} & & & & & \geq & -\hat{u}^* \\
 & & & & & \Delta\hat{v} & & & \geq & -\hat{v}^*
\end{array}
$$

5 Computational Results

We discuss two portfolios in this section, a nine-security one due to Markowitz [21], and a portfolio consisting of the thirty stocks in the Dow Jones Industrial Average. Solving the problem (QP) for different values of E_0 will give a transaction cost efficient frontier (TCEF). The relationship of the TCEF to the efficient frontier with no transaction costs is explored in this section, as well as other properties of the TCEF.

We investigated the Markowitz 9-security portfolio with all transaction costs equal to 5% and the initial portfolio equally divided among the securities. For the return and risk data, see Braun [7]. Figure 2 shows which securities are involved at each point along the TCEF. The vertical scale is the minimum required expected return (ie. the horizontal coordinate) of the efficient frontier plots.

Figure 2 contains a great deal of useful information which can be accessed by visual inspection. For example, looking down a column, changes occurring in the portfolio as the required expected return is decreased can easily be seen. Also, horizontal comparisons between panels allow for comparisons to be made between the no-cost and c=5% frontiers. Note that the highest returning portfolio on the no-cost frontier has no counterpart on the c=5% TCEF. Likewise, the least risky-least returning point on the c=5% TCEF has no counterpart on the no-cost frontier. This is a manifestation of the leftward shift seen in Figure 1. With no transaction costs the highest returning portfolio is always full investment in the single highest yielding individually efficient security. This is security #5 for the Markowitz dataset. However with the introduction of nonzero transaction costs, that is no longer the case.

A general observation is that the portfolios along the TCEF are not simply related to the portfolios along the no transaction cost efficient frontier. Sometimes, entirely new securities are involved. Sometimes, buy and sell decisions are reversed. So Figure 2 highlights that the introduction of costs changes the portfolio rebalancing problem dramatically and that the optimal solutions are also quite different.

We applied our solution strategy to the problem of rebalancing portfolios composed of the 30 stocks which currently make up the Dow Jones Industrials Average. All securities were involved initially, with proportions varying from 1% to 5%. The

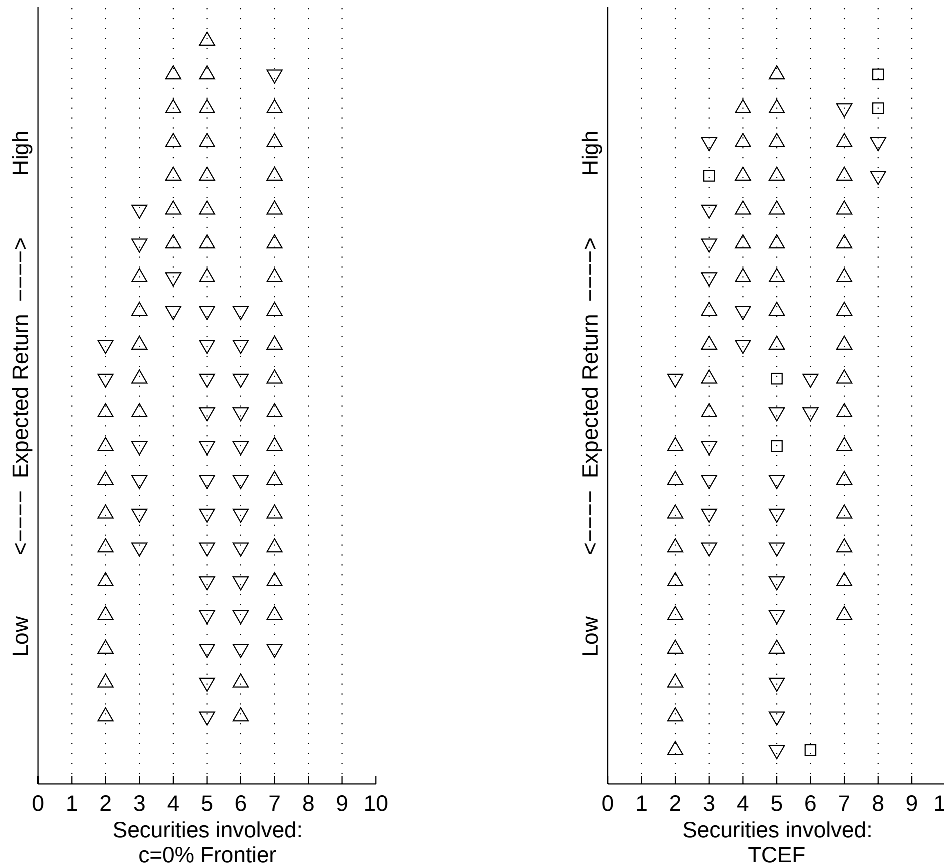

Figure 2: These panels show which securities are involved along the no-transaction cost efficient frontier (left) and the c=5% TCEF (right). If no symbol is present, then a security was liquidated. The symbols $\triangle$ and $\triangledown$ represent buy and sell decisions that did not result in liquidation of the security. The symbol $\square$ represents a security in which the position is unchanged.

buying and selling costs varied from security to security, from 0% to 5%. For the values of the parameters $\bar{x}, c_B, c_S$ as well as for the return and risk data, see Braun [7]. In Figures 3 and 4, we present visualizations of the optimal portfolios for the no transaction cost efficient frontier and the TCEF.

6 Conclusions

The results of this paper will allow the incorporation of transaction costs into portfolio optimization problems, in a manner that leads to intuitive and sensible allocations. The model calculates the risk of the resulting portfolio, weighted by the amount invested after paying transaction costs. The model can be formulated as a quadratic programming problem of size comparable to the model with no transaction costs, so

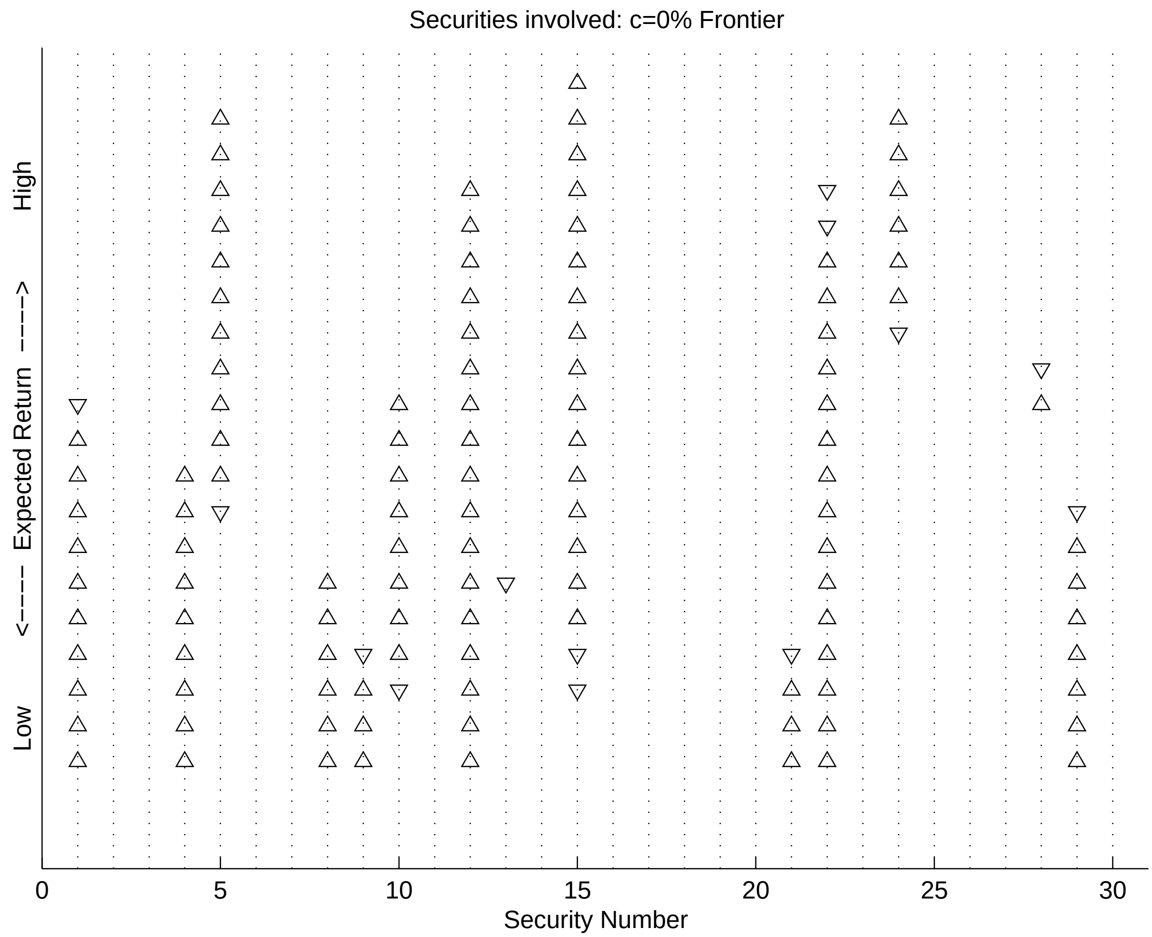

Figure 3: Involved Securities for the Dow Jones Portfolio: This figure shows which securities are involved along the c=0% frontier.

it can be solved equally efficiently.

References

[1] C. J. Adcock and N. Meade. A simple algorithm to incorporate transactions costs in quadratic optimization. *European Journal of Operational Research*, 79(1):85–94, 1994.

[2] S. Albeverio, L. J. Lao, and X. L. Zhao. On-line portfolio selection strategy with prediction in the presence of transaction costs. *Mathematical Methods of OR*, 54:133–161, 2001.

[3] C. Bengtsson and J. Holst. On portfolio selection: Improved covariance matrix estimation for Swedish asset returns. Technical report, Department of Economics,

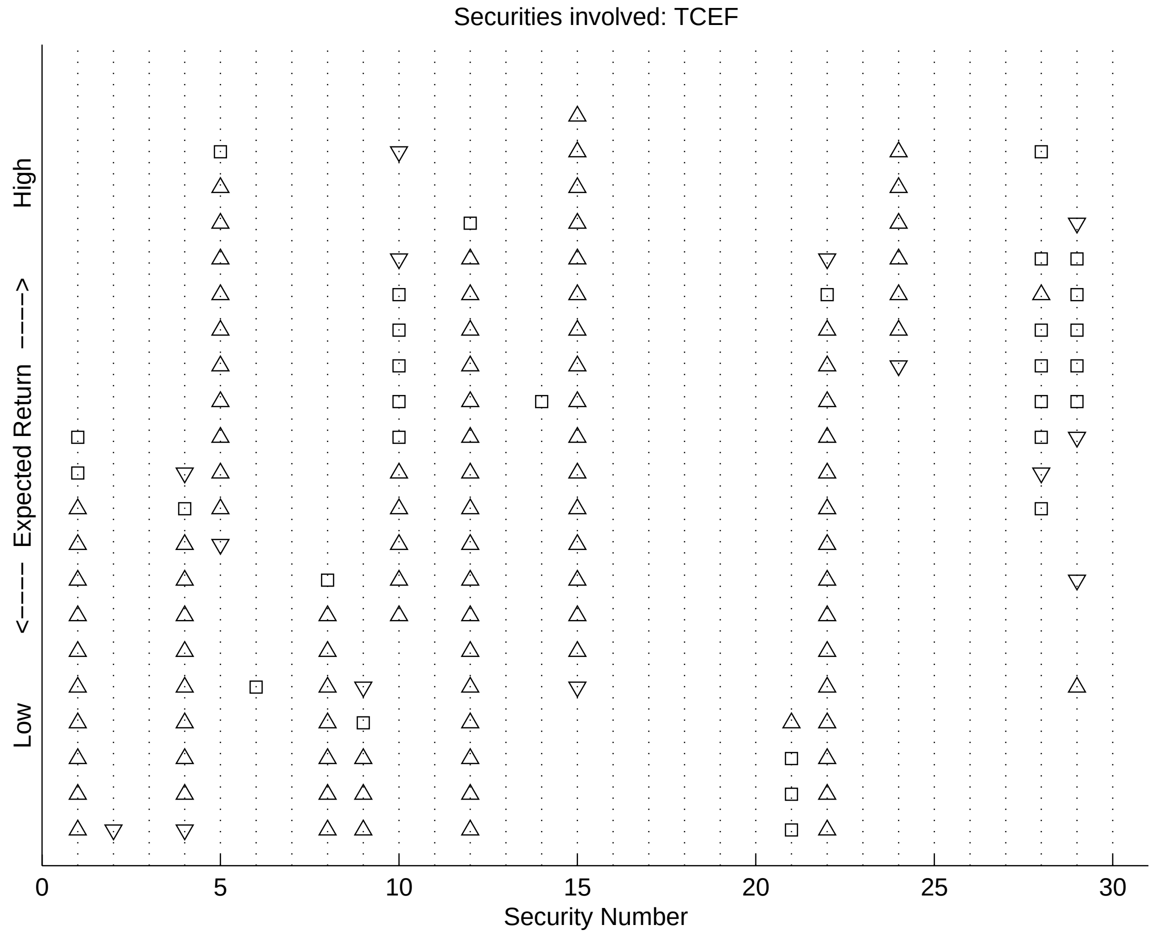

Figure 4: Involved Securities for the Dow Jones Portfolio: This figure shows which securities are involved along the TCEF. Here, overlay comparisons with Figure 3 are possible.

Lund University, Sweden, October 2002.

[4] A. B. Berkelaar, C. Dert, B. Oldenkamp, and S. Zhang. A primal-dual decomposition-based interior point approach to two-stage stochastic linear programming. *Operations Research*, 50(5):904–915, 2002.

[5] D. Bertsimas, C. Darnell, and R. Soucy. Portfolio construction through mixed-integer programming at Grantham, Mayo, Van Otterloo and Company. *Interfaces*, 29(1):49–66, 1999.

[6] D. Bienstock. Computational study of a family of mixed-integer quadratic programming problems. *Mathematical Programming*, 74:121–140, 1996.

[7] S. E. Braun. *Solving a quadratic programming problem subject to orthogonality constraints.* PhD thesis, Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY, December 2001.

[8] T.-J. Chang, N. Meade, J. E. Beasley, and Y. M. Sharaiha. Heuristics for cardinality constrained portfolio optimisation. *Computers and Operations Research*, 27:1271–1302, 2000.

[9] A. Charnes and W. W. Cooper. Programming with linear fractional functionals. *Naval Research Logistics Quarterly*, 9:181–186, 1962.

[10] V. K. Chopra and W. T. Ziemba. The effect of errors in means, variances, and covariances on optimal portfolio choice. In W. T. Ziemba and J. M. Mulvey, editors, *Worldwide asset and liability modeling*, pages 53–61. Cambridge University Press, 1998.

[11] A. Consiglio and S. Zenios. Designing portfolios of financial products via integrated simulation and optimization models. *Operations Research*, 47(2):195–208, 1999.

[12] O. L. V. Costa and A. C. Paiva. Robust portfolio selection using linear-matrix inequalities. *Journal of Economic Dynamics and Control*, 26:889–909, 2002.

[13] M. H. A. Davis and A. R. Norman. Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15:676–713, 1990.

[14] S.-E. Fleten, K. Høyland, and S. W. Wallace. The performance of stochastic dynamic and fixed mix portfolio models. *European Journal of Operational Research*, 140(1):37–49, 2002.

[15] J. Gondzio and R. Kouwenberg. High-performance computing for asset-liability management. *Operations Research*, 49(6):879–891, 2001.

[16] N. Gulpinar, B. Rustem, and R. Settergren. Multistage stochastic mean-variance portfolio analysis with transaction costs. Technical report, Computing, Imperial College, London, July 2001.

[17] J. H. Kamin. Optimal portfolio revision with a proportional transaction cost. *Management Science*, 21(11):1263–1271, 1975.

[18] H. Konno and A. Wijayanayake. Portfolio optimization problem under concave transaction costs and minimal transaction unit constraints. *Mathematical Programming*, 89(2):233–250, 2001.

[19] E. K. Lee and J. E. Mitchell. Computational experience of an interior point SQP algorithm in a parallel branch-and-bound framework. In H. Frenk *et al.*, editor, *High Performance Optimization*, chapter 13, pages 329–347. Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[20] H. Markowitz. Portfolio selection. *Journal of Finance*, 7:77–91, 1952.

[21] H. Markowitz. *Portfolio selection: efficient diversification of investments*. Blackwell, New York, 2nd edition, 1991.

[22] H. Markowitz. Efficient portfolios, sparse matrices, and entities: a retrospective. *Operations Research*, 50(1):154–160, 2002.

[23] R. C. Merton. Lifetime portfolio selection under uncertainty: the continuous time case. *The Review of Economics and Statistics*, L:247–257, August 1969.

[24] J. M. Mulvey and K. Simsek. Rebalancing strategies for long-term investors. In E. J. Kontoghiorghes, B. Rustem, and S. Siokos, editors, *Computational Methods in Decision-Making, Economics and Finance*, volume 74 of *Applied Optimization*, chapter 6. Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.

[25] A. F. Perold. Large-scale portfolio optimization. *Management Science*, 30:1143–1160, 1984.

[26] B. Rustem and R. Settergren. Robust portfolio analysis. In E. J. Kontoghiorghes, B. Rustem, and S. Siokos, editors, *Computational Methods in Decision-Making, Economics and Finance*, volume 74 of *Applied Optimization*, chapter 5. Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.

[27] M. C. Steinbach. Markowitz revisited: mean-variance models in financial portfolio analysis. *SIAM Review*, 43(1):31–85, 2001.

[28] A. Yoshimoto. The mean-variance approach to portfolio optimization subject to transaction costs. *Journal of the Operations Reseach Society of Japan*, 39(1):99–117, 1996.

[29] S. A. Zenios. Asset/liability management under uncertainty for fixed income securities. *Annals of Operations Research*, 59:77–98, 1995. Reprinted in *World Wide Asset and Liability Modeling*, W. T. Ziemba and J. M. Mulvey (editors), Cambridge University Press, 1998.