Rebalancing an Investment Portfolio in the Presence of Transaction Costs

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Abstract

The inclusion of transaction costs is an essential element of any realistic portfolio optimization. In this paper, we consider an extension of the standard portfolio problem in which linear transaction costs are incurred to rebalance an investment portfolio. The Markowitz framework of mean-variance efficiency is used with costs modelled as a percentage of the value transacted. Each security in the portfolio is represented by a pair of continuous decision variables corresponding to the amounts bought and sold. In order to properly represent the variance of the resulting portfolio, we suggest rescaling by the funds available after paying the transaction costs. This results in a fractional quadratic programming problem. We show that this fractional quadratic programming problem has a structure that allows it to be reformulated as an equivalent quadratic programming problem of size comparable to the model without transaction costs. Theoretically, one way to reduce the measure of risk is to buy and sell the same security, which is not an attractive practical strategy. We show that an optimal solution to the quadratic programming reformulation can always be found that does not simultaneously buy and sell a single security. The results of the paper extend the classical Markowitz model to the case of proportional transaction costs in a natural manner with limited computational cost. Computational results for two empirical datasets are discussed.

Keywords: Portfolio optimization, transaction costs, rebalancing, quadratic programming

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1 Introduction

Constructing a portfolio of investments is one of the most significant financial decisions facing individuals and institutions. A decision-making process must be developed which identifies the appropriate weight each investment should have within the portfolio. The portfolio must strike what the investor believes to be an acceptable balance between risk and reward. In addition, the costs incurred when setting up a new portfolio or rebalancing an existing portfolio must be included in any realistic analysis. In this paper, we show that proportional transaction costs can be incorporated in a manner which makes the resulting optimization problem a quadratic program.

Essentially, the standard portfolio optimization problem is to identify the optimal allocation of limited resources among a limited set of investments. Optimality is measured using a tradeoff between perceived risk and expected return. Expected future returns are based on historical data. Risk is measured by the variance of those historical returns.

When more then one investment is involved, the covariance among individual investments becomes important. In fact, any deviation from perfect positive correlation allows a beneficial diversified portfolio to be constructed. Efficient portfolios are allocations that achieve the highest possible return for a given level of risk. Alternatively, efficient portfolios can be said to minimize the risk for a given level of return. These ideas earned their inventor a Nobel Prize and have gained such wide acceptance that countless references could be cited; however, the original source is Markowitz [16].

One standard formulation of the portfolio problem minimizes a quadratic risk measurement with a set of linear constraints specifying the minimum expected portfolio return, E_0, and enforcing full investment of funds. The decision variables x_i are the proportional weights of the i^{th} security in the portfolio. Here n securities are under consideration. We assume the presence of a risk-free security with return ρ, and we let y denote the amount invested in this security. Additionally, μ is the column vector of expected returns and Q is the positive semidefinite covariance matrix. We assume that short selling is not allowed, so the proportions x_i are restricted to be nonnegative. This formulation is:

$$\begin{array}{llcrl} \min & & & \frac{1}{2}x^TQx & \\ \text{s.t.} & \rho y & + & \mu^T x & \geq E_0 \\ & y & + & e^T x & = 1 \\ & & & y, x & \geq 0, \end{array} \tag{1}$$

where e denotes the vector of all ones.

By varying the parameter E_0 and solving multiple instances of this problem, the set of efficient portfolios can be generated. This set, visualized in a risk/return plot, is called the efficient frontier. An investor may decide where along the efficient frontier (s)he finds an acceptable balance between risk and reward.

In this paper, we describe a method for finding an optimal portfolio when proportional transaction costs have to be paid. These costs vary linearly with the amount of a security bought or sold. Our method requires the solution of a quadratic program of similar size to the Markowitz model. If E_0 is very small, it may be necessary to subsequently solve a linear programming problem to ensure that there is not simultaneous buying and selling in a single security. Our method allows different costs for different securities, and different costs for buying and selling. Our model captures the feature that transaction costs are paid when a security is bought or sold and the transaction cost reduces the amount of that particular security that is available. In particular, both the risk and the return in our model are measured using the portfolio arising after paying the transaction costs.

The portfolio rebalancing problem has similarities to the index tracking problem [1, 7, 11]. See Zenios [23] for a discussion of portfolio optimization models. The optimal solution to the portfolio optimization problem is sensitive to the data Q and μ, so estimating this data accurately is the subject of current research; see Chopra and Ziemba [9] or Bengtsson and Holst [2] for example. Stochastic programming approaches to portfolio optimization have been investigated in [10, 12, 19, 20] and elsewhere; such approaches work with sets of scenarios.

Modification of a portfolio should be performed at regular intervals, and determination of the appropriate interval in the presence of transaction costs is a problem of interest. Preferably, selection of the interval should be done in conjunction with selection of the method used for rebalancing. This paper contains a method for rebalancing. There has been interest in portfolios that can be modified continuously, starting with Merton [18]. These methods are generally limited to problems with a small number of securities. For a recent survey on the impact of transaction costs on the dynamic rebalancing problem, see Cadenillas [5]. For a discussion of handling capital gains taxes in dynamic portfolio allocation problems, see Cadenillas and Pliska [6].

This paper is organized as follows. We turn to the portfolio rebalancing problem in §2. First, we motivate the cost model and provide examples of costs that fit this model before introducing an essential change of variables. Alternative approaches to this problem are also discussed in §2. Our model is a quadratic programming problem, and the nature of the solutions to that problem is discussed in §3. If the return requirement is low, the nature of the solutions can change; this is the subject of §4. Computational results for two empirical datasets are presented in §5. Finally, we offer concluding remarks in §6.

2 Portfolio Rebalancing Problem

What we consider is an extension of the basic portfolio optimization problem in which transaction costs are incurred to rebalance a portfolio, $\bar{x}$, into a new and efficient portfolio, x. A portfolio may need to be rebalanced periodically simply as updated risk and return information is generated with the passage of time. Further, any alteration to the set of investment choices would necessitate a rebalancing decision of this type.

In addition to the obvious cost of brokerage fees/commissions, here are two examples of other transaction costs that can be modeled in this way:

1. Capital gains taxes are a security-specific selling cost that can be a major consideration for the rebalancing of a portfolio. For more discussion of the impact of capital gains, especially in a dynamic portfolio allocation model, see Cadenillas and Pliska [6].

2. Another possibility would be to incorporate an investor's confidence in the risk/return forecast as a subjective "cost". Placing high buying and selling costs on a security would favor maintaining the current allocation $\bar{x}$. Placing a high selling cost and low buying cost could be used to express optimism that a security may outperform its forecast.

Let u_i and v_i represent the amount bought and sold (respectively) of risky security i. The amount invested in each of the securities will be

$$x = \bar{x} + u - v. \tag{2}$$

We assume proportional transaction costs. Let c_{B_i} and c_{S_i} denote the transaction cost of buying and selling one unit of risky security i, respectively. We assume $0 \leq c_B < e$, $0 \leq c_S < e$ and $c_S + c_B > 0$. We assume that there are no transaction costs associated with the risk-free security — if there are such costs, we can instead treat this security in the same manner as the other securities. We let x_0 denote the total amount spent on transaction costs, so

$$x_0 = c_B^T u + c_S^T v. \tag{3}$$

The total amount invested in the securities, after paying transaction costs, will be $1 - x_0$. We obtain the constraint

$$y + e^T x = 1 - c_B^T u - c_S^T v.$$

The resulting model for minimizing the variance of the resulting portfolio subject to meeting an expected return of $E_0 > 0$ in the presence of proportional transaction costs is

$$\begin{aligned}
\min \quad & \tfrac{1}{2} x^T Q x && && (4)\\
\text{s.t.} \quad & \rho y + \mu^T x && \geq E_0 && (5)\\
& x - u + v && = \bar{x} && (6)\\
& y + e^T x + c_B^T u + c_S^T v && = 1 && (7)\\
& u, v, x, y \geq 0. && && (8)
\end{aligned}$$

A user might also require restrictions such as limiting the proportion of assets that can be invested in a group of securities. We can express this as a homogeneous constraint on x. For example, if security 1 must constitute no more than 10% of the resulting portfolio, we can impose the constraint

$$9x_1 - \sum_{i=2}^{n} x_i \leq 0.$$

We generalize this to allow m homogeneous constraints in our model, written in the form $ay + Ax \leq 0$ where A is an $m \times n$ matrix and a is an m-vector.

To this point, we have been optimizing the standard risk measure for efficient frontiers, that is:

$$\frac{1}{2} x^T Q x.$$

When there are no transaction costs to be paid, one dollar is always available for investment, i.e. $(y + \sum_{i=1}^{n} x_i = 1)$. This assumption is implicit in the standard risk measure. However, for nonzero transaction costs that implicit assumption is no longer valid. One dollar is not available for investment, costs will be paid to rebalance. The appropriate objective is therefore

$$f(x) := \frac{\frac{1}{2} x^T Q x}{(y + e^T x)^2}. \tag{9}$$

Here x_0 is again the amount paid in transaction costs. Therefore $(1 - x_0)$ is the actual amount available for investment, so we are choosing to scale the standard risk measurement by the square of the dollar amount actually invested.

This gives the fractional quadratic programming problem (FQP) which we will solve to find the optimal portfolio for a given expected return.

$$\begin{array}{llllllll}
\min & & & \frac{x^T Q x}{2(y+e^T x)^2} & & & & \\
\text{s.t.} & \rho y & + & \mu^T x & & & \geq & E_0 \\
(FQP) & & & x & - & u + v & = & \bar{x} \\
& y & + & e^T x & + & c_B^T u + c_S^T v & = & 1 \\
& ay & + & Ax & & & \leq & 0 \\
& & & & & u, v, x, y & \geq & 0.
\end{array}$$

Analytically, notice that with zero transaction costs then $y + e^T x = 1$ and we recover the standard risk measurement. So our choice does pass the first test required of any theoretical extension; recover the previous result. This choice also makes dimensional sense given the quadratic numerator.

Our choice of this fractional objective function also makes intuitive sense. For nonzero transaction costs, there are conflicting effects at work within the portfolio. For a given $\bar{x}$, the absolute amount of principal available for investment will decrease as the transaction cost percentage is increased. But in order to get the same payoff $(\mu^T x)$ on a smaller amount of principal the investor will need to reach for higher returns. This should correlate to taking on higher levels of risk. Our fractional choice effectively boosts the risk measurement for these transaction cost depleted portfolios.

Without the denominator in the objective function, it is possible to obtain optimal solutions which involve both buying and selling a particular security, so both $u_i > 0$ and $v_i > 0$ for some security i. Of course, in practice this is not a desirable strategy. Nonetheless, it does reduce the measure of risk $\frac{1}{2}x^TQx$. The incidence of solutions of this form is particularly noticeable for low values of E_0. Normalizing by the amount invested in securities prevents this undesirable outcome, as we shall see.

The fractional objective $f(x)$ can be made quadratic using the technique of replacing the denominator by the square of the reciprocal of a variable. This is a straightforward extension of the technique of Charnes and Cooper [8] for fractional programs where the objective is a ratio of linear functions and the constraints are linear. Let

$$t := \frac{1}{y + e^T x} \tag{10}$$

and then define

$$\hat{u} := tu, \quad \hat{v} := tv, \quad \hat{x} := tx, \quad \hat{y} := ty. \tag{11}$$

Note that since u and v are constrained to be nonnegative, we must have $t \geq 1$. The constraints (5)–(7) can be multiplied through by t. We also need to include the constraint $\hat{y} + e^T\hat{x} = 1$, which is equivalent to (10). Thus, the fractional quadratic program (FQP) is equivalent to the quadratic programming problem (QP):

$$\begin{array}{llcccccccccc}
\min_{\hat{x},\hat{u},\hat{v},t} & & & \frac{1}{2}\hat{x}^TQ\hat{x} & & & & & & & & \\
\text{s.t.} & \rho\hat{y} & + & \mu^T\hat{x} & & & & & - & E_0t & \geq & 0 \\
(QP) & & & \hat{x} & - & \hat{u} & + & \hat{v} & - & \bar{x}t & = & 0 \\
& \hat{y} & + & e^T\hat{x} & + & c_B^T\hat{u} & + & c_S^T\hat{v} & - & t & = & 0 \\
& \hat{y} & + & e^T\hat{x} & & & & & & & = & 1 \\
& a\hat{y} & + & A\hat{x} & & & & & & & \leq & 0 \\
& & & & & & & \hat{x},\hat{u},\hat{v},t & \geq & 0. & &
\end{array}$$

Once we find a solution $(\hat{y}^*, \hat{x}^*, \hat{u}^*, \hat{v}^*, t^*)$ to (QP), we can obtain a solution (y^*, x^*, u^*, v^*) to the original problem (FQP) by rescaling $\hat{y}$, $\hat{x}$, $\hat{u}$ and $\hat{v}$, so $y^* = \frac{1}{t^*}\hat{y}^*$, $x^* = \frac{1}{t^*}\hat{x}^*$, $u^* = \frac{1}{t^*}\hat{u}^*$, and $v^* = \frac{1}{t^*}\hat{v}^*$.

The efficient frontier is found by optimizing (QP) for different values of E_0. In Figure 1, we graph the efficient frontier for a nine-security problem, with three different choices for the transaction costs, namely zero costs, 3% costs for each buy and sell decision, and 5% costs for each buy and sell decision. The initial portfolio is equally weighted in the nine securities.

The optimal objective function is plotted against the value of E_0. Optimizing this quadratic program creates the situation where the $c = 0\%$ frontier extends furthest into the risk/return plane. Other transaction cost efficient frontiers, abbreviated TCEF, are pulled back from that limit as seen in Figure 1. It is apparent that transaction costs reduce the range of investment choice.

The model (QP) is the one that we propose to solve in order to find an optimal level of risk for a given level of return in the presence of proportional transaction costs. In the next two sections we examine properties of the solutions to (QP). First,

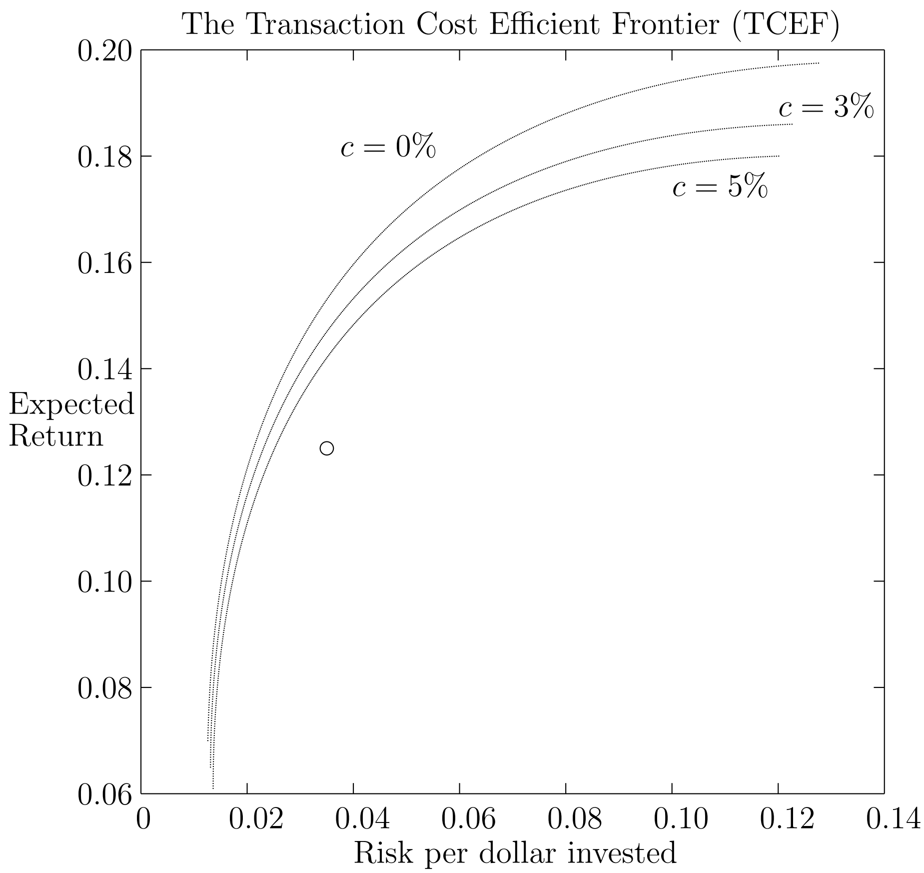

Figure 1: The initial portfolio is located by a circle. Notice that as the level of transaction costs c increases, the curves shift right. Increased transaction costs reduce investment choice.

we conclude this section by considering other methods for portfolio rebalancing that have been discussed in the literature.

Adcock and Meade [1] add a linear term for the costs to the original Markowitz quadratic risk term and minimize this quantity. This requires finding an appropriate balance between the transaction costs and the risk. The model assumes a fixed rate of transaction costs across securities. The risk is measured in terms of the adjusted portfolio before transaction costs are paid. Konno and Wijayanayasake [14] consider a cost structure that is considerably more involved than ours, with the result that the model is harder to solve. Yoshimoto [22] considers a similar transaction cost model to ours and proposes a nonlinear programming algorithm to solve the problem and their computational results indicate that ignoring transaction costs can result in inefficient portfolios.

An alternative model is to reduce the vector of expected returns μ by the transaction costs. The method implicitly assumes that transaction costs are paid at the end of the period, impacting both the risk and the return. If the transaction costs must be paid at the beginning of the period then care must be taken in the sale of assets to pay the transaction costs, in order to ensure that the resulting portfolio has securities in the same proportion. Further, the return calculation assumes a return on the amount paid in transaction costs, so this constraint needs to be modified.

If the only transaction cost is a fixed cost per transaction, one modeling approach is to place an upper bound on the number of transactions. This gives rise to a quadratically constrained integer programming problem. This approach has been investigated widely; see, for example, Perold [21], Bienstock [3] or Lee and Mitchell [15]. The presence of the integrality restriction makes the formulation far harder to solve than the one presented in this paper. An alternative approach is to place these transaction costs directly into the objective, which again results in a quadratic integer programming formulation; see for example Konno and Wijayanayasake [14] or Kellerer et al [13].

3 Optimal solutions to (QP)

We have introduced variables to both buy u and sell v each security. We have not imposed an explicit constraint requiring that if a certain security is bought then it cannot also be sold. Both buying and selling a security would not be a desirable strategy in practice, but it might decrease the risk measure $\hat{x}^T Q \hat{x}$. We call a solution $(\hat{u}, \hat{v})$ *complementary* if it satisfies $\hat{u}^T \hat{v} = 0$, that is, if no stock is both bought and sold. In this section, we show that if the return constraint $\rho y + \mu^T \hat{x} - E_0 t \geq 0$ is active at the optimal solution to (QP) then the optimal solution must be complementary.

If the return constraint is not active at the optimal solution, then it is possible that an optimal solution will not be complementary. We discuss this situation in §4, where we also show that a complementary solution can always be found efficiently even in this situation.

The Karush-Kuhn-Tucker (KKT) optimality conditions for the quadratic program

(QP) require that $(\hat{y}, \hat{x}, \hat{u}, \hat{v}, t)$ be primal feasible and that the following equations can be satisfied:

$$
\begin{aligned}
z + \lambda + a^T \pi - w\rho &= 0 && (12) \\
(z+\lambda)e + A^T \pi + Q\hat{x} - w\mu + s^r - s^x &= 0 && (13) \\
-s^r - zc_B - s^u &= 0 && (14) \\
s^r - zc_S - s^v &= 0 && (15) \\
-wE_0 - \bar{x}^T s^r + z &= 0 && (16) \\
w(\mu^T \hat{x} - E_0 t) &= 0 && (17) \\
\hat{x}^T s^x &= 0 && (18) \\
\hat{u}^T s^u &= 0 && (19) \\
\hat{v}^T s^v &= 0 && (20) \\
\pi^T (a\hat{y} + A\hat{x}) &= 0. && (21)
\end{aligned}
$$

Here, w is a nonnegative scalar, z and λ are free scalars, π is a nonnegative m-vector, s^r is a free n-vector, and s^x, s^u, and s^v are nonnegative n-vectors. We have exploited the fact that $t \geq 1$ in any feasible solution in deriving (16).

Assume that $\hat{u}_i > 0$ and $\hat{v}_i > 0$ for some security i. We argue that we must then have $w = 0$ for the KKT conditions to hold. If $\hat{u}_i > 0$ and $\hat{v}_i > 0$, it follows from (19) and (20) that $s^u_i = s^v_i = 0$. Adding together the ith components of (14) and (15) gives

$$z(c_{Bi} + c_{Si}) = 0.$$

Since $c_S + c_B > 0$, we immediately obtain $z = 0$. Now adding together the whole of (14) and (15) gives $s^u + s^v = 0$, so $s^u = s^v = 0$ and $s^r = 0$. Substituting into (16) for s^r and z and gives $wE_0 = 0$. Thus, we have proved the following theorem.

Theorem 1 *If the optimal solution to (QP) has a strictly positive Karush-Kuhn-Tucker multiplier w for the return constraint $\rho y + \mu^T \hat{x} - E_0 t \geq 0$ then the optimal solution is complementary.*

If there is no risk-free security then the theorem still holds. The proof given above did not involve the equations impacted by y, so it is still valid.

4 Low return requirements

If an investor just wishes to minimize risk, with little concern for expected return, then the value of E_0 can be set to a low number, with the result that the return constraint may be satisfied strictly in the solution to (QP). Alternatively, it may be that the presence of the homogeneous constraints $ay + Ax \leq 0$ results in an optimal solution that satisfies the return constraint strictly. If the return constraint is not active at the optimal solution, then an optimal solution may not be complementary.

Nonetheless, there is an optimal solution that is complementary, and that solution can be found efficiently, as we argue in this section.

Let $(\hat{y}^*, \hat{x}^*, \hat{u}^*, \hat{v}^*, t^*)$ be an optimal solution to (QP). Assume this point is a noncomplementary solution. If we modify $\hat{u}$, $\hat{v}$, and t while fixing $\hat{y} = \hat{y}^*$ and $\hat{x} = \hat{x}^*$ then the objective function value in (QP) does not change. Reducing t will reduce the amount spent in transaction costs. This suggests solving the following linear programming problem:

$$
\begin{array}{llcrcrcl}
\min & & & & & t & & \\
\text{s.t.} & -c_B^T \hat{u} & - & c_S^T \hat{v} & + & t & = & \hat{y}^* + e^T \hat{x}^* \\
(LP^t) & \hat{u} & - & \hat{v} & + & \bar{x} t & = & \hat{x}^* \\
 & \hat{u} & & & & & \geq & 0 \\
 & & & \hat{v} & & & \geq & 0
\end{array}
$$

Examining the dual of this linear program and the LP complementary slackness conditions shows that the optimal solution must satisfy $\hat{u}^T \hat{v} = 0$, in a similar manner to the proof of Theorem 1. The structure of this LP is such that an efficient iterative scheme can be developed to find the optimal solution.

5 Computational Results

We discuss two portfolios in this section, a nine-security one due to Markowitz [17], and a portfolio consisting of the thirty stocks in the Dow Jones Industrial Average. Solving the problem (QP) for different values of E_0 will give a transaction cost efficient frontier (TCEF).

We investigated the Markowitz 9-security portfolio with all transaction costs equal to 3% and with all transaction costs equal to 5%, with the initial portfolio equally divided among the securities in both cases. The results are plotted in Figure 1. For the return and risk data, and more details of the results, see Braun [4]. A general observation is that the portfolios along the TCEF are not simply related to the portfolios along the no transaction cost efficient frontier. Sometimes, entirely new securities are involved. Sometimes, buy and sell decisions are reversed. The introduction of costs changes the portfolio rebalancing problem dramatically and the optimal solutions are also quite different.

We applied our solution strategy to the problem of rebalancing portfolios composed of the 30 stocks which currently make up the Dow Jones Industrials Average. All securities were involved initially, with proportions varying from 1% to 5%. The buying and selling costs varied from security to security, from 0% to 5%. For more details on these experiments, see Braun [4]. As with the earlier example, the optimal solution was altered by the transaction costs.

6 Conclusions

The results of this paper will allow the incorporation of transaction costs into portfolio optimization problems, in a manner that leads to intuitive and sensible allocations. The model calculates the risk of the resulting portfolio, weighted by the amount invested after paying transaction costs. The model can be formulated as a quadratic programming problem of size comparable to the model with no transaction costs, so it can be solved equally efficiently.

From the computational results presented in this paper, it appears that the effect of transaction costs is more marked for relatively high levels of desired expected return, since the portfolio manager is then forced to perform a lot of rebalancing because only a few assets can meet the desired return requirement.

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