Approximation Algorithms from Inexact Solutions to Semidefinite Programming Relaxations of Combinatorial Optimization Problems☆

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Abstract

Semidefinite relaxations of certain combinatorial optimization problems lead to approximation algorithms with performance guarantees. For large-scale problems, it may not be computationally feasible to solve the semidefinite relaxations to optimality. In this paper, we investigate the effect on the performance guarantees of an approximate solution to the semidefinite relaxation for MaxCut, Max2Sat, and Max3Sat. We show that it is possible to make simple modifications to the approximate solutions and obtain performance guarantees that depend linearly on the most negative eigenvalue of the approximate solution, the size of the problem, and the duality gap. In every case, we recover the original performance guarantees in the limit as the solution approaches the optimal solution to the semidefinite relaxation.

Key words: semidefinite programming, approximation algorithms, MaxCut, Max2Sat, Max3Sat

1. Introduction

A celebrated randomized algorithm for the combinatorial optimization problems of Max-Cut and Max2Sat was developed by Goemans and Williamson [1]. This algorithm exploited a semidefinite programming (SDP) relaxation of each combinatorial optimization problem in order to derive a performance guarantee. In particular, they proved that their algorithm is an α-approximation algorithm with $\alpha = 0.878$ for MaxCut, so their algorithm returns a solution with value at least 0.878 of the optimal value. Stronger results were subsequently derived for Max2Satand Max3Sat.

Semidefinite programs can be solved using primal-dual interior point methods in polynomial time [2]. Recently, there has been theoretical and some practical interest in even stronger

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relaxations, such as completely positive relaxations [3, 4]. In this paper, we look at weakening the semidefinite relaxation, which is sometimes necessary for larger scale problems. In practice, the interior point approach can be slow and may not converge in an acceptable time for large-scale problems. Therefore, there has been interest in using alternative methods to solve semidefinite programs approximately, for example in [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. These methods may relax the linear constraints or the semidefinite constraints.

In this paper, we investigate how the Goemans-Williamson bound and its extensions are affected by the use of an approximate solution. If the semidefiniteness constraint is relaxed in either the primal or dual problem, we obtain characterizations of the bound based on the most negative eigenvalue of the corresponding matrix. When the algorithm is terminated before optimality is reached, we obtain characterizations of the bound based on the duality gap. These two characterizations can be combined for algorithms that relax both positive semidefiniteness and a tolerance on duality.

We will take the following as our standard primal SDP problem with an $n \times n$ matrix variable X and with m linear constraints:

$$
\begin{aligned}
\max_X \quad & C \bullet X \\
\text{subject to} \quad \mathbf{A}X &= b \\
X &\in \mathcal{S}_n.
\end{aligned}
\tag{1}
$$

Here, C and X are $n \times n$ symmetric matrices, $C \bullet X$ denotes the Frobenius product of C and X, $\mathcal{S}_n$ denotes the set of symmetric positive semidefinite $n \times n$ matrices, the notation $\mathbf{A}X$ represents an m-vector with ith component equal to $A^i \bullet X$ with A^i an $n \times n$ symmetric matrix, and b is an m-vector. Throughout this paper, the SDP relaxation of the combinatorial optimization problem under consideration will have the form (1), and we will find a feasible solution to the combinatorial optimization problem by first finding a feasible solution to (1) and then applying a rounding procedure to this feasible solution. A bound on the quality of the solution to the combinatorial optimization problem can be obtained through the use of a feasible solution to the SDP dual of (1), namely:

$$
\begin{aligned}
\min_{y,S} \quad & b^T y \\
\text{subject to} \quad \mathbf{A}^T y - S &= C \\
S &\in \mathcal{S}_n
\end{aligned}
\tag{2}
$$

where $y \in \mathbb{R}^m$ and $\mathbf{A}^T y = \sum_{i=1}^m y_i A^i$. The approximation bounds that we obtain require taking primal and dual variables (X, y, S) returned by an algorithm, using them to construct feasible solutions to (1) and (2), and then deriving a bound on the ratio of the objective function values of these feasible solutions.

The semidefinite program (1) can be relaxed as in [10], for example. This relaxation has the form

$$
\begin{aligned}
\max \quad & C \bullet X \\
\text{subject to} \quad \mathbf{A}X &= b \\
X &\in T_n \supseteq \mathcal{S}_n
\end{aligned}
\tag{3}
$$

where T_n is a convex cone. This formulation includes as special cases linear programming relaxations of (1) as in [9, 16], second order cone and low dimensional cone relaxations as

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in [17, 13, 14], and versions of the spectral bundle method [18, 19, 7]. Algorithmically, the cone T_n is modified as the algorithm proceeds, leading to tighter relaxations. If we let T_n^* be the dual cone then the dual problem of (3) is

$$
\begin{array}{rlll}
\min_{y,S} & b^T y & & \\
\text{subject to} & \mathbf{A}^T y - S & = & C \\
& S & \in & T_n^* \subseteq \mathcal{S}_n,
\end{array}
\tag{4}
$$

a constrained version of (2). Relaxations of this form for MAXCUT and MAX2SAT are considered in §2.1 and §3.1, respectively. By relaxing the primal problems of the original MAXCUT, MAX2SAT and MAX3SAT, we acquire indefinite matrices X. At the same time, we are constraining the corresponding dual problems, so any feasible dual solutions can produce upper bounds on both the original combinatorial problem and on the SDP relaxation of MAXCUT, MAX2SAT and MAX3SAT. By exploiting the duality gap and the values we get from those indefinite matrices and α-approximation, we can develop an approximation algorithm to find a ratio between the values of our relaxed version of the original SDP for MAXCUT, MAX2SAT and MAX3SAT and the actual optimal values of MAXCUT, MAX2SAT and MAX3SAT. Our techniques exploit the fact that the identity matrix is feasible in the primal SDP relaxation of MAXCUT, which allows construction of feasible solutions that are convex combinations of the identity matrix and an indefinite matrix. The identity matrix is not feasible in the relaxations of MAX2SAT and MAX3SAT that we consider, so we exploit a related feasible solution for these problems.

An analogous relaxation of the dual problem (2) can also be constructed by relaxing the requirement that $S \in \mathcal{S}_n$, and this is considered in §2.2 and §3.2 for MAXCAT and MAX2SAT, respectively. The primal and dual relaxation results are extended to MAX3SAT in §4. Finally, in §5 we consider relaxing the equality constraints as well as the positive semidefiniteness constraints.

2. The MAXCUT Problem

The MAXCUT problem on a graph $G = (V, E)$ with $n := |V|$ and with edge weights w_e $\forall e \in E$ is to partition the vertices V into two sets V^1 and V^2 so as to maximize

$$
Z(V^1, V^2) \quad := \sum_{(i,j) \in E \,:\, i \in V^1, j \in V^2} w_{ij}.
$$

We let Z_{mc}^* denote the optimal value of the MAXCUT problem. The SDP relaxation of the MAXCUT problem is

$$
\begin{array}{rlll}
\max & C \bullet X & & \\
\text{subject to} & X_{ii} & = & 1 \qquad i = 1, \ldots, n \\
& X & \in & \mathcal{S}_n
\end{array}
\tag{5}
$$

where the matrix $C = \frac{1}{4} L$ with L equal to the Laplacian matrix $\text{Diag}\,(We) - W$ with each entry w_{ij} of W is the weight of the edge (i, j) and e denotes the vector of ones. The optimal

value of the semidefinite program is denoted Z_{SDP}^{mc*} and provides an upper bound on the optimal value of the MAXCUT problem. If each $w_e \geq 0$ then the optimal solution to this SDP can then be converted into a feasible solution to the MAXCUT problem in such a way that the ratio of the two values is no smaller than 0.878 [1], so

$$Z_{SDP}^{mc*} \geq Z_{mc}^* \geq 0.878 Z_{SDP}^{mc*}. \tag{6}$$

More precisely, the performance ratio is closer to 0.87856, and it should be noted that an SDP may have an irrational solution, and (6) can be modified to reflect this tolerance. For simplicity, we will express the ratio as 0.878 in the remainder of this paper. The primal and dual SDP relaxations of MAXCUT both have strictly feasible solutions, so strong duality holds. Any feasible solution $\hat{X}$ can be converted into a feasible solution to the MAXCUT problem, as indicated in the following lemma.

Lemma 1. *[1] Assume all the edge weights are nonnegative. Let $\hat{X}$ be a feasible solution to (5) with value $\hat{Z} := C \bullet \hat{X}$. There exists a partition of V satisfying $Z(V^1, V^2) \geq 0.878\hat{Z}$.*
□

The proof is constructive, using a rounding procedure to get a feasible solution to the original MAXCUT problem with value at least $0.878(C \bullet \hat{X})$. In this paper, we construct weakenings of (6) when problem (5) is only solved approximately.

2.1. MAXCUT primal approach

Conceptually, we consider an algorithm of the following form:

Conceptual algorithm for MAXCUT

Construct an approximation of the form (3) to (5).
Let $\hat{X}$ be a feasible solution to the approximation, so $\hat{X}_{ii} = 1$ for $i = 1, \ldots, n$.
Let $(\hat{y}, \hat{S})$ be feasible in the dual problem (4). Define $\Delta := b^T \hat{y} - C \bullet \hat{X}$.
Modify $\hat{X}$ to give a feasible solution $\bar{X}$ to (5).
Use $\bar{X}$ to construct a cut (V^1, V^2) with value $Z_{mc} \geq 0.878C \bullet \bar{X}$.

The aim is then to obtain a lower bound on the ratio of Z_{mc} to Z_{mc}^*, providing a performance guarantee for the algorithm. Let $\lambda_{min}(\hat{X}) < 0$ be the minimum eigenvalue of $\hat{X}$. Then we can choose

$$\bar{X} := (1 - \theta)I + \theta\hat{X}, \tag{7}$$

which is feasible in (5) if $\theta \leq \hat{\theta} := \frac{1}{1 - \lambda_{min}(\hat{X})} < 1$. If $\lambda_{min}(\hat{X}) \geq 0$ then we can take $\theta = 1$. The rounding procedure of Lemma 1 can be applied to $\bar{X}$ to get a feasible solution to MAXCUT, with a guaranteed approximation ratio.

Proposition 1. *Assume all the edge weights are nonnegative. The performance ratio of the conceptual algorithm for* MAXCUT *is bounded below as follows:*

$$\frac{Z_{mc}}{Z_{mc}^*} \geq 0.878 \frac{\theta C \bullet \hat{X} + (1 - \theta) C \bullet I}{C \bullet \hat{X} + \Delta}.$$

PROOF. From Lemma 1, we have $Z_{mc} \geq 0.878 C \bullet \bar{X}$. Further, Z_{mc}^* is bounded above by $b^T \hat{y}$. Thus, we have

$$\frac{Z_{mc}}{Z_{mc}^*} = \frac{Z_{mc}}{C \bullet \bar{X}} \frac{C \bullet \bar{X}}{Z_{mc}^*} \geq 0.878 \frac{\theta C \bullet \hat{X} + (1 - \theta) C \bullet I}{C \bullet \hat{X} + \Delta}$$

as required. □

Now suppose $\hat{X}$ is the optimal solution to the relaxed SDP of MAXCUT, so the duality gap $\Delta = 0$. The result in the Proposition can be strengthened as in the following theorem.

Theorem 2. *Assume all the edge weights are nonnegative. Assume $\hat{X}$ is the optimal solution to (3) and $\hat{X} \notin \mathcal{S}_n$. Choose $\theta = \hat{\theta}$. Then*

$$\frac{Z_{mc}}{Z_{mc}^*} \geq 0.878(0.5\hat{\theta} + 0.5).$$

PROOF. The duality gap $\Delta = 0$ and $C \bullet \hat{X} \geq Z_{mc}^*$. Then from Proposition 1, we conclude

$$\frac{Z_{mc}}{Z_{mc}^*} = \frac{Z_{mc}}{C \bullet \bar{X}} \frac{C \bullet \bar{X}}{Z_{mc}^*} \geq 0.878 \frac{\hat{\theta} C \bullet \hat{X}}{Z_{mc}^*} + 0.878 \frac{\left(1 - \hat{\theta}\right) C \bullet I}{Z_{mc}^*} \geq 0.878\hat{\theta} + 0.878 \frac{\left(1 - \hat{\theta}\right) C \bullet I}{Z_{mc}^*}.$$

Now

$$C \bullet I = \frac{1}{4} \left(Diag\left(We\right) - W\right) \bullet I = \frac{1}{4} \sum_{i,j} w_{ij} = \frac{1}{2} \sum_{i<j} w_{ij} \geq \frac{1}{2} Z_{mc}^*$$

so

$$\frac{Z_{mc}}{Z_{mc}^*} \geq 0.878\hat{\theta} + 0.878\frac{1}{2}\left(1 - \hat{\theta}\right)$$

and the result follows. □

Note that we recover the original result of Goemans and Williamson [1] in the limit as $\lambda_{\min}(\hat{X}) \nearrow 0$.

Now we will examine MAXCUT with general edge weights (so w_{ij} can be negative). For several practical MAXCUT problems, edge weights can take a negative value. In this situation, Nesterov [20] showed that a rounding procedure can give a solution with value

$$Z_{SDP}^{mc*} \geq Z_{mc}^* \geq \frac{2}{\pi} Z_{SDP}^{mc*}.$$

However, our approach cannot exploit this result because $L \bullet I$ is not necessarily larger than Z_{mc}^*, or even nonnegative. Thus we will examine a slightly different representation of the MaxCut problem. Let W^-, W^+ represent the sums of all the negative and positive edges respectively, and we assume $W^- < 0 < W^+$. Let $\bar{X}$ be feasible in (5). It is shown in [1] that $\bar{X}$ can be rounded to a partition satisfying

$$Z_{mc} - W^- = Z(V^1, V^2) - W^- \geq 0.878\,(C \bullet \bar{X} - W^-).$$

Using (7) and the fact that

$$\frac{Z^{mc} - W^-}{Z^{ub} - W^-} = \frac{Z^{mc} - W^-}{Z^{feas} - W^-} \cdot \frac{Z^{feas} - W^-}{Z^{ub} - W^-}$$

we conclude

$$\frac{Z^{mc} - W^-}{Z^{ub} - W^-} \geq 0.878 \left(\frac{\theta Z^{ub} + \frac{1}{2}\,(1 - \theta) \sum_{i<j} w_{ij} - W^-}{Z^{ub} - W^-} \right) \tag{8}$$

Proposition 2. *Assume $\hat{X}$ is the optimal solution to (3) and $\hat{X} \notin \mathcal{S}_n$. Choose $\theta = \hat{\theta}$. Then*

$$\frac{Z_{mc} - W^-}{Z_{mc}^* - W^-} \geq 0.878(0.5\hat{\theta} + 0.5).$$

PROOF. We have

$$
\begin{aligned}
\frac{Z_{mc} - W^-}{Z_{mc}^* - W^-} &= \frac{Z_{mc} - W^-}{C \bullet \bar{X} - W^-} \frac{C \bullet \bar{X} - W^-}{Z_{mc}^* - W^-} \\[2mm]
&\geq 0.878 \frac{\hat{\theta} C \bullet \hat{X} + \left(1 - \hat{\theta}\right) C \bullet I - W^-}{C \bullet \hat{X} - W^-} \\[2mm]
&\geq 0.878\hat{\theta} + 0.878 \frac{\left(1 - \hat{\theta}\right)(C \bullet I - W^-)}{Z_{mc}^* - W^-}.
\end{aligned}
$$

Now

$$C \bullet I = \frac{1}{4}\left(Diag\,(We) - W\right) \bullet I = \frac{1}{4}\sum_{i,j} w_{ij} = \frac{1}{2}\sum_{i<j} w_{ij} \geq \frac{1}{2}(Z_{mc}^* + W^-)$$

so

$$\frac{Z_{mc} - W^-}{Z_{mc}^* - W^-} \geq 0.878\hat{\theta} + 0.878\frac{1}{2}\left(1 - \hat{\theta}\right)$$

and the result follows. $\qquad\square$

2.2. MaxCut *dual approach*

In this section, we consider relaxing the positive semidefiniteness on the dual slack matrix. The dual to the SDP relaxation of a MaxCut problem (5) is

$$\begin{array}{rrcl} \min & & e^T y & \\ \text{subject to} & \text{diag}\,(y) - S & = & C \\ & S & \in & \mathcal{S}_n \end{array} \qquad (9)$$

where e denotes the vector of ones. A conic relaxation of this takes the form

$$\begin{array}{rrcl} \min & & e^T y & \\ \text{subject to} & \text{diag}\,(y) - S & = & C \\ & S & \in & T_n \supseteq \mathcal{S}_n \end{array} \qquad (10)$$

for a convex cone T_n, with the primal relaxation being correspondingly constrained:

$$\begin{array}{rrcll} \max & & C \bullet X & & \\ \text{subject to} & X_{ii} & = & 1 & i = 1, \ldots, n \\ & X & \in & T_n^* \subseteq \mathcal{S}_n. & \end{array} \qquad (11)$$

Such an approach to MaxCut is considered in [21], for example. Below we give an analogue to Proposition 1 when (10) is solved approximately; Theorem 2 can be extended similarly when (10) is solved to optimality.

Proposition 3. *Assume all the edge weights are nonnegative. Let $(\hat{y}, \hat{S})$ be feasible in (10) and $\hat{X}$ be feasible in (11), with duality gap $\Delta = e^T \hat{y} - C \bullet \hat{X}$. Assume $\lambda_{\min}(\hat{S}) < 0$. Then a partition can be constructed from $\hat{X}$ with value Z_{mc} satisfying*

$$\frac{Z_{mc}}{Z_{mc}^*} \geq 0.878 \frac{e^T \hat{y} - \Delta}{e^T \hat{y} + n \left| \lambda_{\min}(\hat{S}) \right|}$$

PROOF. We can construct a feasible solution to (9) by setting

$$\bar{y} = \hat{y} + e \left| \lambda_{min}(\hat{S}) \right|$$

and so the corresponding dual slack matrix is

$$\bar{S} = \hat{S} + \left(\left| \lambda_{min}\left(\hat{S} \right) \right| \right) I \succeq 0.$$

Then $Z_{mc}^* \leq e^T \bar{y}$ so we obtain

$$\frac{Z_{mc}}{Z_{mc}^*} = \frac{Z_{mc}}{C \bullet \hat{X}} \frac{C \bullet \hat{X}}{Z_{mc}^*} \geq 0.878 \frac{e^T \hat{y} - \Delta}{e^T \hat{y} + n \left| \lambda_{min}(\hat{S}) \right|}$$

as required. $\qquad \qquad \square$

The original result of [1] is recovered in the limit when (10) is solved to optimality and $\hat{S} \in \mathcal{S}_n$.

3. The MAX2SAT Problem

We can use a similar approaches to the MAX2SAT problem. The MAX2SAT problem consists of boolean variables $x_1, x_2, ..., x_n$ and a set C of clauses c_{ij} . Each clause c_{ij} is made up at most two distinct literals from x_i, x_j, and their complements $\bar{x}_i$ and $\bar{x}_j$. The problem asks us to find the truth assignment to each x_i to maximize the number of satisfied clauses or to maximize the sum of the weights of the satisfied clauses, where the weights are all positive. Let Z^*_{m2s} denote the optimal value of the MAX2SAT problem. An SDP relaxation of MAX2SAT was originally presented in [1], based on expressing MAX2SAT as a quadratic integer program and lifting as for the MAXCUT program. This was improved by Feige and Goemans [22] by expanding the size of the matrix, adding additional linear constraints to the SDP, and modifying the rounding procedure. It was further improved by Lewin et al. [23] through an additional modification of the rounding procedure.

In the formulation of [22], each literal is included explicitly, so we have ± 1 variables t_i corresponding to the original boolean variables and also ± 1 variables f_i corresponding to the complements of the original boolean variables. We write $y \in \mathbb{R}^{2n}$ to represent these two vectors, so $y := (t^T, f^T)^T$. Triangle inequalities are also added, corresponding to the valid constraints

$$y_i y_j + y_i y_k + y_j y_k \geq -1, \quad 1 \leq i, j, k \leq 2n \tag{12}$$

for each distinct choice of i, j, and k, and

$$y_i y_j + y_i + y_j \geq -1, \quad 1 \leq i, j \leq 2n \tag{13}$$

for each distinct choice of i and j. Let w_{ij} correspond to the clause with literals corresponding to y_i and y_j, where we allow $i = j$. The formulation in [22] is then

$$
\begin{aligned}
\max_{y,Y} \quad & \tfrac{1}{4} \sum w_{ij} \left(3 + y_i + y_j - Y_{ij}\right) \quad := g(y, Y) \\
\text{subject to} \quad & Y_{ii} = 1 && i = 1, \ldots, 2n \\
& y_i + y_{n+i} = 0 && i = 1, \ldots, n \\
& Y_{(n+i)j} = -Y_{ij} && i = 1, \ldots, n,\ j = 1 \ldots, n \\
& Y_{i(n+j)} = -Y_{ij} && i = 1, \ldots, n,\ j = 1 \ldots, n \\
& Y_{(n+i)(n+j)} = Y_{ij} && i = 1, \ldots, n,\ j = 1 \ldots, n \\
& A_l \bullet Y \geq -1 && \text{for } l \in \mathcal{L} \\
& \begin{pmatrix} 1 & y^T \\ y & Y \end{pmatrix} \in \mathcal{S}_{2n+1}
\end{aligned}
\tag{14}
$$

where $\mathcal{L}$ denotes the set of additional constraints. It was proved in [22] that this formulation results in a 0.931 approximation algorithm for MAX2SAT. It is shown in [23] that this formulation allows a 0.940-approximation algorithm for MAX2SAT, through the use of a modified rounding procedure. Further, any feasible solution (y, Y) to (14) with value $g(y, Y)$ can be rounded to a feasible solution to the MAX2SAT problem with value

$$Z_{m2s} \geq 0.940 \, g(y, Y).$$

We introduce some more notation in order to ease the subsequent presentation. In particular, we introduce a symmetric $(2n + 1) \times (2n + 1)$ matrix Γ and a symmetric $n \times n$ matrix M satisfying

$$\Gamma \quad := \quad \begin{pmatrix} 1 & y^T \\ y & Y \end{pmatrix} \quad = \quad \begin{pmatrix} 1 & t^T & -t^T \\ t & M & -M \\ -t & -M & M \end{pmatrix}. \tag{15}$$

3.1. MAX2SAT *Primal approach*

Our relaxation of (14) is

$$
\begin{array}{rlll}
\max_{y,Y} & \frac{1}{4} \sum w_{ij} \left(3 + y_i + y_j - Y_{ij}\right) \\
\text{subject to} & Y_{ii} = 1 & i = 1, \dots, 2n \\
& y_i + y_{n+i} = 0 & i = 1, \dots, n \\
& Y_{(n+i)j} = -Y_{ij} & i = 1, \dots, n, \; j = 1 \dots, n \\
& Y_{i(n+j)} = -Y_{ij} & i = 1, \dots, n, \; j = 1 \dots, n \\
& Y_{(n+i)(n+j)} = Y_{ij} & i = 1, \dots, n, \; j = 1 \dots, n \\
& A_l \bullet Y \geq -1 & \text{for } l \in \mathcal{L} \\
& \begin{pmatrix} 1 & y^T \\ y & Y \end{pmatrix} \in T_{2n+1} \supseteq \mathcal{S}_{2n+1}
\end{array} \tag{16}
$$

for a convex cone T_{2n+1}. Let $(\hat{y}, \hat{Y})$ be a feasible solution to (16), define $\hat{\Gamma}$, $\hat{M}$, and $\hat{t}$ using (15), and assume the minimum eigenvalue $\lambda_{\min}(\hat{\Gamma}) < 0$. Let

$$\tilde{I} := \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \in \mathcal{S}_{2n} \quad \text{and} \quad \hat{I} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & -I \\ 0 & -I & I \end{pmatrix} \in \mathcal{S}_{2n+1} \tag{17}$$

and define

$$\bar{\Gamma} := \theta \hat{\Gamma} + (1 - \theta) \hat{I}. \tag{18}$$

Note that $\bar{\Gamma}$ satisfies the linear constraints of (16) provided $0 \leq \theta \leq 1$.

Lemma 3. *The matrix $\bar{\Gamma}$ is positive semidefinite provided*

$$0 \leq \theta \leq \hat{\theta} := \frac{1}{1 - \lambda_{\min}(\hat{\Gamma})} < 1.$$

PROOF. Let $u, v \in \mathbb{R}^n$ and $u_0 \in \mathbb{R}$. Let $r = 0.5(u - v)$ and $s = 0.5(u + v)$. We have

$$
\begin{pmatrix} u_0 \\ u \\ v \end{pmatrix}^T \bar{\Gamma} \begin{pmatrix} u_0 \\ u \\ v \end{pmatrix} = \left(\begin{pmatrix} u_0 \\ r \\ -r \end{pmatrix} + \begin{pmatrix} u_0 \\ s \\ s \end{pmatrix} \right)^T \bar{\Gamma} \left(\begin{pmatrix} u_0 \\ r \\ -r \end{pmatrix} + \begin{pmatrix} u_0 \\ s \\ s \end{pmatrix} \right)
$$

$$
= \begin{pmatrix} u_0 \\ r \\ -r \end{pmatrix}^T \bar{\Gamma} \begin{pmatrix} u_0 \\ r \\ -r \end{pmatrix}
$$

$$
= \begin{pmatrix} u_0 \\ r \\ -r \end{pmatrix}^T \left(\theta \, \hat{\Gamma} + (1 - \theta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & -I \\ 0 & -I & I \end{pmatrix} \right) \begin{pmatrix} u_0 \\ r \\ -r \end{pmatrix}
$$

$$
\geq \quad \theta \, \lambda_{\min}(\hat{\Gamma}) \, (u_0^2 + 2r^T r) + (1 - \theta) \, (u_0^2 + 4r^T r)
$$

$$
\geq \quad (1 - \theta \, (1 - \lambda_{\min}(\hat{\Gamma}))) \, (u_0^2 + 2r^T r)
$$

$$
\geq \quad 0 \quad \forall u_0, u, v
$$

as required. $\qquad \square$

It follows that the matrix $\bar{\Gamma}$ and the corresponding $(\bar{y}, \bar{Y})$ are feasible in (14).

Theorem 4. *Assume all the clause weights w_{ij} are nonnegative. Let $(\hat{y}, \hat{Y})$ be an optimal solution to (16) with corresponding matrix $\hat{\Gamma}$. Define $\bar{\Gamma}$ using (18) with $\theta = \hat{\theta}$, define $(\bar{y}, \bar{Y})$ from (15), and round $\bar{\Gamma}$ using the method in [23] to give a feasible solution to the MAX2SAT problem with value Z_{m2s}. Then*

$$
\frac{Z_{m2s}}{Z_{m2s}^*} \geq 0.940 \, (\frac{1}{2}\hat{\theta} + \frac{1}{2}).
$$

If every clause contains two distinct literals and if no clause is a tautology then the ratio can be improved to

$$
\frac{Z_{m2s}}{Z_{m2s}^*} \geq 0.940 \, (\frac{1}{4}\hat{\theta} + \frac{3}{4}).
$$

PROOF. We have

$$
\frac{Z_{m2s}}{Z_{m2s}^*} = \frac{Z_{m2s}}{g(\bar{y}, \bar{Y})} \frac{g(\bar{y}, \bar{Y})}{Z_{m2s}^*}
$$

$$
\geq \quad 0.940 \, \frac{\hat{\theta} g(\hat{y}, \hat{Y}) + (1 - \hat{\theta}) g(0, \tilde{I})}{Z_{m2s}^*}
$$

$$
\geq \quad 0.940 \, \hat{\theta} + 0.940 \, \frac{(1 - \hat{\theta}) 0.5 \sum w_{ij}}{Z_{m2s}^*}
$$

$$
\text{since } Z_{m2s}^* \leq g(\hat{y}, \hat{Y}), \text{ and } 3 + y_i + y_j - Y_{ij} \geq 2 \, \forall i, j \text{ if } y = 0, Y = \tilde{I}
$$

$$
\geq \quad 0.940 \hat{\theta} + 0.940 \frac{1}{2}(1 - \hat{\theta}) \qquad \text{since } \sum w_{ij} \geq Z_{m2s}^*
$$

10

as required. If every clause contains two distinct literals and if no clause is a tautology then

$$g(0, \tilde{I}) = \frac{3}{4} \sum w_{ij}$$

and the result follows. □

In the limit as $\lambda_{\min}(\hat{\Gamma}) \nearrow 0$ we recover the result of [23].

3.2. MAX2SAT *Dual approach*

The dual problem to (14) has an objective function of

$$\min h(\gamma, \pi) := \frac{3}{4} \sum w_{ij} + e^T \gamma - e^T \pi \tag{19}$$

where the free dual variables $\gamma \in \mathbb{R}^{1+2n}$ correspond to the constraints $Y_{ii} = 1$ for $i = 1, \ldots, 2n$ and the constraint that the top left entry in Γ is equal to 1, and the nonpositive variables π have entries π_l for each $l \in \mathcal{L}$. The dual slack matrix is denoted Φ and satisfies

$$\Phi = \mathrm{diag}(\gamma) + \Upsilon, \tag{20}$$

where Υ is a linear function of the remaining dual variables, namely π and the variables corresponding to the homogeneous linear primal constraints. We relax the positive semidefiniteness constraint on Φ to

$$\Phi \in T_{2n+1} \supseteq S_{2n+1} \tag{21}$$

so the positive semidefiniteness constraint on Γ is tightened to $\Gamma \in T^*_{2n+1} \subseteq S_{2n+1}$. We have the following proposition.

Proposition 4. *Assume $\hat{\gamma}$ and $\hat{\pi}$ satisfy the linear constraints in the dual problem for some choice of the remaining dual variables corresponding to the homogeneous linear primal constraints, with the corresponding slack matrix $\hat{\Phi} \in T_{2n+1} \setminus S_{2n+1}$. Assume there is a solution $(\hat{y}, \hat{Y})$ satisfying the linear constraints of (14) with the corresponding $\hat{\Gamma} \in T^*_{2n+1}$ and let Δ denote the duality gap. Let Z_{m2s} be the value of the solution obtained by rounding $\hat{\Gamma}$. Then*

$$\frac{Z_{m2s}}{Z^*_{m2s}} \geq 0.940 \left(1 - \frac{\Delta - (2n+1)\lambda_{\min}(\hat{\Phi})}{\frac{3}{4} \sum w_{ij} + e^T \hat{\gamma} - e^T \hat{\pi} - (2n+1)\lambda_{\min}(\hat{\Phi})} \right)$$

where $\frac{3}{4} \sum w_{ij} + e^T \hat{\gamma} - e^T \hat{\pi} - (2n+1)\lambda_{\min}(\hat{\Phi})$ provides an upper bound on Z^*_{m2s}.

PROOF. By taking

$$\bar{\gamma} = \hat{\gamma} - \lambda_{\min}(\hat{\Phi}) e \quad \text{and} \quad \bar{\Phi} = \hat{\Phi} - \lambda_{\min}(\hat{\Phi}) I \succeq 0,$$

we obtain to a dual feasible solution to (14), so $\frac{3}{4} \sum w_{ij} + e^T \hat{\gamma} - e^T \hat{\pi} - (2n+1)\lambda_{\min}(\hat{\Phi})$ provides an upper bound on Z_{m2s}^*. We have

$$
\begin{aligned}
\frac{Z_{m2s}}{Z_{m2s}^*} &= \frac{Z_{m2s}}{g(\hat{y}, \hat{Y})} \frac{g(\hat{y}, \hat{Y})}{Z_{m2s}^*} \\
&\geq 0.940 \frac{\frac{3}{4} \sum w_{ij} + e^T \hat{\gamma} - e^T \hat{\pi} - \Delta}{\frac{3}{4} \sum w_{ij} + e^T \hat{\gamma} - e^T \hat{\pi} - (2n+1)\lambda_{\min}(\hat{\Phi})}
\end{aligned}
$$

as required. □

We recover the result of [22] in the limit as $\Delta \to 0$ and $\lambda_{\min}(\hat{\Phi}) \nearrow 0$.

4. The MAX3SAT Problem

In the MAX3SAT problem, each clause contains at most 3 literals and has nonnegative weight w_{ijk}. We consider two approaches in this section. In §4.1, we examine reductions of MAX3SAT to MAX2SAT and exploit our results in §3. In §4.2, we look at a more direct approach to MAX3SAT. We let Z_{m3s}^* denote the optimal value of the instance of MAX3SAT.

4.1. Using gadgets

Gadgets are techniques used to reduce one problem to another. They are formally defined in [24] and they are explored by Trevisan et al. [25]. For an instance of MAX3SAT, a clause of length 3 can be replaced by a set of clauses of length 2 each with some weight, and it is desired that the original clause is satisfiable if and only if the sum of the weights of the satisfied new clauses is equal to some threshold. More formally, we have the following definition:

Definition 1. *A* **strict α-gadget** *is a construction that maps a clause $l_1 \vee l_2 \vee l_3$ in three literals into a collection of clauses of length no more than two involving the original literals, their complements, and auxiliary boolean variables, with the properties that*

1. *if the original clause is satisfied then there is an assignment of the auxiliary variables such that the sum of the weights of the satisfied new clauses is equal to α, and there is no assignment of the auxiliary variables that has the sum of the weights of the satisfied new clauses strictly greater than α, and*
2. *if the original clause is unsatisfied then there is an assignment of the auxiliary variables such that the sum of the weights of the satisfied new clauses is equal to $\alpha-1$, and there is no assignment of the auxiliary variables that has the sum of the weights of the satisfied new clauses strictly greater than $\alpha - 1$.*

Garey et al. [26] proposed a strict 7-gadget for MAX3SAT, with each original 3-clause replaced by 10 clauses of length 2. More recently, Trevisan et al. [25], proposed a strict 3.5-gadget reducing method, with a clause of length 3

$$(x_i \vee x_j \vee x_h)$$

replaced by 7 different clauses length 2, namely

$$x_i \vee x_h, \bar{x}_i \vee \bar{x}_h, x_i \vee \bar{y}_{ijh}, \bar{x}_i \vee \bar{y}_{ijh}, x_h \vee \bar{y}_{ijh}, \bar{x}_h \vee y_{ijh}, x_j \vee y_{ijh}$$

with the first six clauses having one-half of the original weight and the last one having the original weight. The auxilary variable y_{ijh} is specific to the clause. Any truth assignment which makes the original length-3 clause true and any truth assignment to y_{ijh}, the sum of the weights of the new 7 clauses will not exceed 3.5. In addition, for any truth assignment which makes the original length-3 clause true, there exists an assignment of y_{ijh} so that the sum of the weights of the new 7 clauses is exactly equal to 3.5. Also for any truth assignment which makes the original length-3 clause false and any truth assignment to y_{ijh}, the sum of the weight of the new 7 clauses will not exceed 2.5. Finally for any truth assignment which makes the original length-3 clause false, there exists a truth assignment to y_{ijh} so that the sum of the weights of the new 7 clauses is exactly 2.5. It was shown in [25] that one can derive an 0.801-approximation for MAX3SAT by exploiting the strict 3.5 gadget. More explicitly, they have the following theorem:

Theorem 5. *[25, Lemma 6.3] Asume there exists a strict 3.5 gadget for* MAX3SAT *and a β-approximation algorithm for* MAX2SAT*. Then there exists a ρ-approximation algorithm for* MAX3SAT *with*

$$\rho = \frac{1}{2} + \frac{(\beta - 1/2)(3/8)}{2.5(1 - \beta) + (3/8)}.$$

The result in [25] is actually more general, allowing clauses of length 1 and 2 to be handled differently, resulting in a stronger result. For our purposes, we will not exploit this generalization. The following theorem is a direct consequence of Theorem 4.

Theorem 6. *Let $(\hat{y}, \hat{Y})$ along with $\hat{\Gamma}$ be the optimal solution to the relaxation of the form (16) of the SDP relaxation of the* MAX2SAT *reformulation of an instance of* MAX3SAT *constructed by applying the strict 3.5 gadget. Let $\lambda_{\min}(\hat{\Gamma}) < 0$ be the minimum eigenvalue of $\hat{\Gamma}$, let $\hat{\theta} = 1/(1 - \lambda_{\min}(\hat{\Gamma}))$, and let $\beta = 0.940(1/2)(1 + \hat{\theta})$. A feasible solution to the instance of* MAX3SAT *can be constructed with value Z_{m3s} satisfying*

$$\frac{Z_{m3s}}{Z_{m3s}^*} \geq \frac{1}{2} + \frac{(\beta - 1/2)(3/8)}{2.5(1 - \beta) + (3/8)}.$$

Further, if there are no unit clauses and if there are no tautologies then this ratio can be improved by taking $\beta = 0.940(1/4)(3 + \hat{\theta})$.

4.2. A direct approach to MAX3SAT primal

Karloff and Zwick [27] constructed a 7/8-approximation algorithm for MAX3SAT by modifying the objective function of (14). In their analysis, the triangle constraints (12) and

(13) were not necessary, so they constructed the formulation

$$
\begin{aligned}
\max_{y,Y,z} \quad & \sum w_{ijk} z_{ijk} \\
\text{subject to} \quad z_{ijk} &\leq \frac{4+y_j+y_k-Y_{ij}-Y_{ik}}{4} && \forall 1 \leq i,j,k \leq 2n \\
z_{ijk} &\leq \frac{4+y_i+y_k-Y_{ij}-Y_{jk}}{4} && \forall 1 \leq i,j,k \leq 2n \\
z_{ijk} &\leq \frac{4+y_i+y_j-Y_{ik}-Y_{jk}}{4} && \forall 1 \leq i,j,k \leq 2n \\
z_{ijk} &\leq 1 && \forall 1 \leq i,j,k \leq 2n \\
Y_{ii} &= 1 && i = 1,\ldots,2n \\
y_i + y_{n+i} &= 0 && i = 1,\ldots,n \\
Y_{(n+i)j} &= -Y_{ij} && i = 1,\ldots,n,\ j = 1.\ldots,n \\
Y_{i(n+j)} &= -Y_{ij} && i = 1,\ldots,n,\ j = 1.\ldots,n \\
Y_{(n+i)(n+j)} &= Y_{ij} && i = 1,\ldots,n,\ j = 1.\ldots,n \\
\begin{pmatrix} 1 & y^T \\ y & Y \end{pmatrix} &\in \mathcal{T}_{2n+1} \supseteq \mathcal{S}_{2n+1}
\end{aligned}
\tag{22}
$$

We define Γ as in (15).

If every clause with positive weight has three distinct literals and is not a tautology then $\Gamma = \hat{I}$ is optimal for any $\mathcal{T}_{2n+1} \supseteq \mathcal{S}_{2n+1}$, with value $\sum w_{ijk}$, since it is feasible to take each $z_{ijk} = 1$. Thus, in this case, we can immediately recover the performance guarantee of [27] by just taking this trivial solution, regardless of the value of Γ.

Now consider the more general case, consisting of clauses of length 1 and/or 2 as well as clauses of length 3. In (22), the shorter clauses correspond to triples (i, j, k) where not all the indices are distinct. With $y = 0$ and $Y = \hat{I}$, we can take

$$
z_{ijk} = \begin{cases} 1 & \text{for clauses of length 3} \\ 0.75 & \text{for clauses of length 2} \\ 0.5 & \text{for clauses of length 1} \end{cases}
\tag{23}
$$

and for this choice we obtain an objective function value that is at least equal to one half of the sum of the clause weights, so it is at least equal to one half of Z^*_{m3s}. Given a feasible solution $(\hat{y}, \hat{Y})$ with corresponding $\hat{\Gamma} \in \mathcal{T}_{2n+1} \setminus \mathcal{S}_{2n+1}$, we produce a feasible solution to the SDP relaxation of MAX3SAT using the construction of (18). The following theorem can then be proved in an analogous manner to Theorem 4.

Theorem 7. *Assume all the clause weights w_{ijk} are nonnegative. Let $(\hat{y}, \hat{Y})$ be an optimal solution to (22) with corresponding matrix $\hat{\Gamma}$. Define $\bar{\Gamma}$ using (18) with $\theta = \hat{\theta} = 1/(1 - \lambda_{\min}(\hat{\Gamma}))$, define $(\bar{y}, \bar{Y})$ from (15), and round $\bar{\Gamma}$ using the method in [27] to give a feasible solution to the MAX2SAT problem with value Z_{m3s}. We have*

$$
\frac{Z_{m3s}}{Z^*_{m3s}} \geq (7/8) \left(\frac{1}{2}\hat{\theta} + \frac{1}{2} \right).
$$

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PROOF. Let $z(y, Y)$ denote the optimal choice of z for a given choice of y and Y in (22). We have

$$
\begin{aligned}
\frac{Z_{m3s}}{Z^*_{m3s}} &= \frac{Z_{m3s}}{\sum w_{ijk} z_{ijk}(\bar{y}, \bar{Y})} \frac{\sum w_{ijk} z_{ijk}(\bar{y}, \bar{Y})}{Z^*_{m3s}} \\
&\geq (7/8) \frac{\hat{\theta} \sum w_{ijk} z_{ijk}(\hat{y}, \hat{Y})) + (1 - \hat{\theta}) \sum w_{ijk} z_{ijk}(0, \hat{I})}{Z^*_{m3s}} \\
&\geq (7/8) \hat{\theta} + (7/8) \frac{(1 - \hat{\theta}) 0.5 \sum w_{ijk}}{Z^*_{m3s}} \\
&\qquad \text{since } Z^*_{m3s} \leq \sum w_{ijk} z_{ijk}(\hat{y}, \hat{Y})), \text{ and from (23)} \\
&\geq (7/8) \hat{\theta} + (7/8) \frac{1}{2}(1 - \hat{\theta}) \qquad \text{since } \sum w_{ijk} \geq Z^*_{m3s}
\end{aligned}
$$

as required. □

The original result of [27] is recovered in the limit as $\lambda_{\min}(\hat{\Gamma}) \nearrow 0$.

5. Approximation Algorithms with Approximate Solution to the Equalities

In the earlier sections, we have examined relaxing the positive semidefiniteness requirements in either the primal problem (1) or the dual problem (2). In this section, we look at more general relaxations of the MAXCUT problem with nonnegative edge weights. In particular, in §5.1 we examine a Lagrangian dual approach where the primal linear constraints are relaxed, and in §5.2 we show how any primal and dual set of variables (X, y, S) can be used to construct a feasible solution to MAXCUT, with a bound on its solution quality.

5.1. A Lagrangian approach to MAXCUT

Iyengar et al. [8] investigated approaches to semidefinite relaxations of combinatorial optimization problems where the linear constraints were further relaxed. This contrasts with the approaches earlier in this paper where we relaxed the positive semidefiniteness constraint on either the matrix of primal variables or the dual slack matrix, but we always satisfied the linear constraints in the primal and dual problems. The approach in [8] was motivated by considering the following equivalent reformulation of the MAXCUT problem (5):

$$
\begin{array}{lll}
\max & C \bullet X & \\
\text{subject to} & X_{ii} \leq 1 & i = 1, \ldots, n \\
& X \succeq 0, \ Tr(X) \leq n &
\end{array} \tag{24}
$$

The equivalence follows because each diagonal entry of C is positive provided the graph contains no isolated vertices, so the inequalities $X_{ii} \leq 1$ hold at equality at optimality. A Lagrangian relaxation of (24) can be constructed as follows:

$$
\zeta(y) = \max_{X \succeq 0, Tr(X) \leq n} \left(C \bullet X + \sum_i y_i(1 - X_{ii}) \right) \tag{25}
$$

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The Lagrangian dual problem is then

$$\min_{y\geq 0} \max_{X\succeq 0, Tr(X)\leq n} (C \bullet X + \sum_i y_i(1 - X_{ii})) \tag{26}$$

The solution method proposed in [8] computes an approximate saddle point of the Lagrangian function and then recovers an ϵ-optimal solution to (24). We denote this ϵ-optimal solution by $\hat{X}$, which is feasible in (24) and within ϵ of optimality of the problem. Then we can find a diagonal matrix with $D_{ii} \geq 0, \forall i$ such that

$$\bar{X} = \hat{X} + D$$

and $\text{diag}(\bar{X}) = I$ which is feasible to the original semidefinite MAXCUT programming problem. We take Z_{mc} to be the value of the feasible solution to the MAXCUT problem obtained by rounding $\bar{X}$.

Theorem 8. *Let Z^*_{mc} be the optimal MAXCUT value. Then*

$$\frac{Z_{mc}}{Z^*_{mc}} \geq 0.878(1 - 2\frac{\epsilon}{\sum_{i<j} w_{ij}})$$

Before we show the proof of the above theorem, we will state a well known lemma (see, for example [28]).

Lemma 9. *Let Z^{mc} be the optimal value of the MAXCUT problem with edge weight w_{ij}. Then*

$$Z^{mc} \geq \frac{1}{2} \sum_{i<j} w_{ij}$$

Now we will show the proof of Theorem 8.

PROOF OF THEOREM 8. We have

$$Z_{mc} \geq 0.878C \bullet \bar{X}$$

and

$$Z^*_{mc} \leq C \bullet \hat{X} + \epsilon.$$

It follows that

$$\begin{aligned}
\frac{Z_{mc}}{Z^*_{mc}} &\geq 0.878\frac{C \bullet \bar{X}}{C \bullet \hat{X} + \epsilon} \\
&= 0.878\frac{C \bullet \hat{X} + C \bullet D}{C \bullet \hat{X} + \epsilon} \\
&\geq 0.878\frac{C \bullet \hat{X}}{C \bullet \hat{X} + \epsilon}
\end{aligned}$$

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But since $C \bullet \hat{X} \geq Z^{mc} - \epsilon$ and the function $g(x) = \frac{x}{x+\epsilon}$ is increasing in x, we conclude

$$\frac{C \bullet \hat{X}}{C \bullet \hat{X} + \epsilon} \geq \frac{Z^{mc} - \epsilon}{Z^{mc} - \epsilon + \epsilon} = 1 - \frac{\epsilon}{Z^{mc}}.$$

Using Lemma 9, we also conclude

$$1 - \frac{\epsilon}{Z^{mc}} \geq 1 - \frac{2\epsilon}{\sum_{i<j} w_{ij}}$$

This implies

$$\frac{Z^{mc}}{C \bullet \hat{X}} \geq 0.878(1 - \frac{2\epsilon}{\sum_{i<j} w_{ij}})$$

as required. $\square$

5.2. An approximation ratio for MAXCUT from any (X, y, S)

In this section, we restrict attention to MAXCUT with nonnegative edge weights. We assume we have a primal matrix $\hat{X}$, a dual vector $\hat{y}$, and a dual slack matrix $\hat{S} := \mathrm{diag}(\hat{y}) - C$. If any diagonal entry of $\hat{X}$ is negative, its objective function value can be improved by setting such an entry equal to zero; this adjustment does not decrease the bound derived below, so we assume that each $\hat{X}_{ii} \geq 0$ for $i = 1, \ldots, n$. We also assume $\hat{X}$ and $\hat{S}$ are symmetric, but make no further assumptions about $(\hat{X}, \hat{y}, \hat{S})$. Such a solution might be returned by a proximal point method or an alternating direction method, as in [12] for example. We want to use $\hat{X}$ to construct a primal feasible solution to the SDP relaxation (5) and hence a feasible solution to the MAXCUT problem, and $(\hat{y}, \hat{S})$ to construct a dual feasible solution to (5) and hence an upper bound on the optimal value of the MAXCUT problem.

A dual feasible solution $\bar{y}$ can be constructed as in the proof of Proposition 3, namely

$$\bar{y} = \hat{y} + e \left| \lambda_{\min}(\hat{S}) \right|,$$

giving an upper bound of $b^T \hat{y} + n \left| \lambda_{\min}(\hat{S}) \right|$.

For the primal solution, we could solve a two variable problem to construct a solution of the form

$$\bar{X} = \beta I + \gamma \hat{X} + \mathrm{diag}(\delta) \tag{27}$$

where $\mathrm{diag}(\delta)$ is a positive semidefinite diagonal matrix and β and γ are nonnegative variables chosen to ensure $\bar{X}$ is feasible in the SDP relaxation of MAXCUT. In particular, we could solve the 2-variable linear program

$$\begin{array}{rlrll}
\max_{\beta,\gamma} & \beta C \bullet I & + & \gamma C \bullet \hat{X} & \\
\text{subject to} & \beta & + & \hat{X}_{ii}\gamma & \leq 1 \quad i = 1, \ldots, n \\
& \beta & + & \lambda_{\min}(\hat{X})\gamma & \geq 0 \\
& & & \beta, \gamma & \geq 0
\end{array} \tag{28}$$

The first n linear constraints ensure each diagonal entry of $\gamma \hat{X}$ is no larger than one. The $(n+1)$th constraint ensures that $\beta I + \gamma \hat{X}$ is positive semidefinite. When each $\hat{X}_{ii} = 1$, the optimal choice for β and γ returns the same $\bar{X}$ as in (7).

Theorem 10. *Let $\hat{X}$ be an $n \times n$ symmetric matrix, let $\hat{y} \in \mathbb{R}^n$, and let $(\hat{\beta}, \hat{\gamma})$ be an optimal solution to (28). Let $\hat{\delta}_i = 1 - \hat{\beta} - \hat{\gamma}\hat{X}_{ii}$ for $i = 1, \ldots, n$. Define $\bar{X}$ as in (27). The matrix $\bar{X}$ is feasible in (5). Rounding $\bar{X}$ gives a feasible solution to MAXCUT with value Z_{mc} satisfying*

$$\frac{Z_{mc}}{Z_{mc}^*} \geq 0.878 \frac{\hat{\beta}C \bullet I + \hat{\gamma}C \bullet \hat{X} + \sum_{i=1}^n C_{ii}\hat{\delta}_i}{b^T\hat{y} + n\left|\lambda_{\min}(\hat{S})\right|}.$$

PROOF. The numerator in the fraction is the value of $C \bullet \bar{X}$ and the denominator is the value of the dual feasible solution $\bar{y}$. The feasibility of $\bar{X}$ follows from construction, and then the 0.878 term follows from [1]. $\square$

Given a symmetric matrix $\hat{X}$, we want to construct a nearby matrix that is feasible in (5). One choice would be to project $\hat{X}$ onto the feasible region of (5). This is the same as finding the nearest correlation matrix in Frobenius norm [29] and is as hard as solving the SDP relaxation [30], so we looked at simpler alternatives above. It might seem attractive to use a sequential projection approach, so first project $\hat{X}$ onto the positive semidefinite cone and then project onto the linear equality constraints. However, there is no guarantee that the resulting matrix will be positive semidefinite, as we illustrate in the following example.

Example 1. *Let*

$$\begin{aligned}
\hat{X} &= \begin{bmatrix} 1 & (\frac{1}{2}+\nu) & -(\frac{1}{2}+\nu) \\ (\frac{1}{2}+\nu) & 1 & (\frac{1}{2}+\nu) \\ -(\frac{1}{2}+\nu) & (\frac{1}{2}+\nu) & 1 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}(\frac{3}{2}+\nu)\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}(\frac{3}{2}+\nu)\begin{bmatrix} 1 & -1 & -2 \end{bmatrix} \\
&\quad + \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}(-2\nu)\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}
\end{aligned}$$

with $\nu > 0$. The projection of a symmetric matrix onto $\mathcal{S}_n$ is obtained by deleting the eigenvectors corresponding to negative eigenvalues [31], giving in this case:

$$\tilde{X} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}(\frac{3}{2}+\nu)\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}(\frac{3}{2}+\nu)\begin{bmatrix} 1 & -1 & -2 \end{bmatrix}.$$

Projection onto the linear inequalities then sets all the diagonal entries equal to one, so the final result is

$$\hat{X} = \begin{bmatrix} 1 & (\frac{1}{2}+\frac{1}{3}\nu) & -(\frac{1}{2}+\frac{1}{3}\nu) \\ (\frac{1}{2}+\frac{1}{3}\nu) & 1 & (\frac{1}{2}+\frac{1}{3}\nu) \\ -(\frac{1}{2}+\frac{1}{3}\nu) & (\frac{1}{2}+\frac{1}{3}\nu) & 1 \end{bmatrix},$$

which is not in $\mathcal{S}_3$: the eigenvector $(1, -1, 1)^T$ has an eigenvalue of $-\frac{2}{3}\nu$. Repeating the sequential projection procedure gives a positive semidefinite matrix in the limit, but the matrix is indefinite at every iteration.

6. Conclusions

SDP relaxations have the best known performance guarantees for several classes of combinatorial optimization problems. Interior point methods can be used to get within ϵ of optimality of a well-conditioned semidefinite program in time polynomial in $\ln(\frac{1}{\epsilon})$ and the size of the problem. However, in practice interior point methods may be too slow to solve the SDP relaxations of many problems, so there has been interest in developing alternative approaches to solving SDPs. In this paper, we have shown that approximate solutions to an SDP relaxation can still be used to obtain a feasible solution to the combinatorial optimization problem with a performance guarantee, through the use of simple modifications of the primal and/or dual solutions. It may be of interest to use more complicated modifications in order to improve the bounds, but as we show in Example 1 iterative approaches such as alternating projection may not lead to feasible solutions to the SDP relaxation.

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