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Using quadratic convex reformulation to tighten the convex relaxation of a quadratic program with complementarity constraints

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Abstract Quadratic Convex Reformulation (QCR) is a technique that has been proposed for binary and mixed integer quadratic programs. In this paper, we extend the QCR method to convex quadratic programs with linear complementarity constraints (QPCCs). Due to the complementarity relationship between the nonnegative variables y and w, a term $y^T Dw$ can be added to the QPCC objective function, where D is a nonnegative diagonal matrix chosen to maintain the convexity of the objective function and the global resolution of the QPCC. Following the QCR method, the products of linear equality constraints can also be used to perturb the QPCC objective function, with the goal that the new QP relaxation provides a tighter lower bound. By solving a semidefinite program, an equivalent QPCC can be obtained whose QP relaxation is as tight as possible. In addition, we extend the QCR to a general quadratically constrained quadratic program (QCQP), of which the QPCC is a special example. Computational tests on QPCCs are presented.

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1 Introduction

Mathematical programs with complementarity constraints (MPCCs) are constrained optimization problems subject to complementarity relations between pairs of nonnegative variables. Extensive applications of MPCCs can be found in hierarchical (particularly bi-level) decision making, inverse optimization, parameter identification, optimal design, and many other contexts; various examples of MPCCs are documented in [6,16]. Convex quadratic programs with complementarity constraints (QPCCs) are one of the prominent subclasses of MPCCs that play the fundamental role in this family of non-convex problems. A convex QPCC can be formulated as

$$\begin{aligned} &\min_{(x,y)} q(x,y) \ := c^T x + d^T y + \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{bmatrix} Q_1 & R \\ R^T & Q_2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &\text{s.t.} \ \ Ax + By = f \\ &\text{and} \ \ 0 \le y \perp w := q + Nx + My \ge 0, \end{aligned} \tag{1}$$

where $x \in I\!R^n$, $y \in I\!R^m$, $A \in I\!R^{k\times n}$, $B \in I\!R^{k\times m}$, $Q := \begin{bmatrix} Q_1 & R \\ R^T & Q_2 \end{bmatrix} \in I\!R^{(n+m)\times(n+m)}$ is positive semidefinite, and $\perp$ is the notation for perpendicularity, which in this context is short hand for the complementarity condition: $y^T w = 0$. When Q is a zero matrix, (1) is a linear program with (linear) complementarity constraints (LPCC). Being a recent entry to the optimization field, the QPCC and its special case of a LPCC have recently been studied in [3,6] wherein a logical Benders scheme [4] was proposed for their global resolution. The complementarity constraints $y \perp w$ render the QPCC to be nonconvex and disjunctive, and the problem is NP-hard. Global resolution schemes for QPCCs work with convex relaxations of (1).

The Quadratic Convex Reformulation (QCR) technique of Billionnet et al. [1,2] was originally proposed for binary quadratic programs, by which the objective function is reformulated using the optimal solution of a semidefinite program (SDP) in such a manner that, the reformulated objective function is convex and the continuous QP relaxation bound is as tight as possible. In this paper, we extend the QCR method to the QPCC problem, with the goal of reformulating the quadratic objective function so that, (1) the global resolution as well as the convexity of the objective function is maintained; (2) the new QP relaxation bound is as tight as possible. The bound is equal to the value of the SDP relaxation of the QPCC, provided a constraint qualification holds. This is also the continuation of the scheme of adding $y^T Dw$, where D is a nonnegative diagonal matrix, to the QPCC objective function to render an equivalent QPCC with a perturbed objective function, as in [3]. As the complementarity constraints are a special form of quadratic constraints, we actually extend the QCR to a general quadratically constrained quadratic program. The advantage of the QP relaxation of the perturbed QCQP versus the SDP lifting of the QCQP is that the QP relaxation is in the space of the original variables.

The present paper is organized as follows: the scheme of adding a nonnegative term $y^T Dw$ to the objective function of a convex QPCC is introduced in §2; extending the QCR method to a general quadratically constrained quadratic program is proposed in §3; the reformulated QPCC is related to its semidefinite relaxation in §4; and computational results tested on convex QPCCs are presented in §5. The example QPCCs are generated following the QPECgen method [9].

2 A simple penalty scheme

In [3], we proposed the addition of a term of the form $y^T Dw$ to the QPCC objective function while maintaining the convexity of the quadratic objective function and the global resolution of the QPCC, where y and w are the complementary variables and D is a nonnegative diagonal matrix. An appropriate matrix D can be found by solving an SDP. The power of this method is illustrated by the following example.

Example 1

$$\begin{aligned} \min_{(y,w)} \; & y^2 + w^2 \\ \text{s.t.} \; & y + w = 1 \\ & 0 \le y \perp w \ge 0. \end{aligned}$$

The above QPCC has an optimal value of 1 whereas its QP relaxation has an optimal value of only 0.5. A new QPCC can be constructed by adding $2yw$ to the quadratic objective function $y^2 + w^2$:

$$\begin{aligned} \min_{(y,w)} \; & (y + w)^2 \\ \text{s.t.} \; & y + w = 1 \\ & 0 \le y \perp w \ge 0. \end{aligned}$$

Both QPCCs are equivalent since y and w are perpendicular to each other; however, the QP relaxation of the new QPCC has an optimal value of 1.

Generalizing the same idea, we could add $y^T Dw$ to the QPCC objective function while maintaining the convexity of the quadratic objective function. The QP relaxation of the new QPCC provides a tighter lower bound. This reshaping of the quadratic objective function exploits the complementarity relationship between the nonnegative variables y and w, while maintaining the convexity of the objective function and global resolution of the QPCC.

It complements the technique of tightening up the relaxation by generating additional valid convex constraints [14]. Therefore we define a linear function $F : \mathcal{D}_+^m \to \mathcal{S}^{n+m}$ by

$$F(D) = \begin{bmatrix} 0 & N^T D \\ DN & M^T D + DM \end{bmatrix} = \sum_{i=1}^m d_i \begin{bmatrix} 0 & N_i^T e_i^T \\ e_i N_i & e_i M_i + M_i^T e_i^T \end{bmatrix} =: \sum_i^m d_i K_i, \tag{2}$$

where d_i is the ith diagonal entry of the nonnegative diagonal matrix D, $\{e_i, i = 1, 2, \cdots, m\}$ is the standard Euclidean basis for $I\!R^m$, and N_i, M_i are the ith rows of matrices N and M respectively. Note as well that

$$y^T D w \;=\; q^T D y \;+\; \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^T F(D) \begin{pmatrix} x \\ y \end{pmatrix}.$$

The new QPCC problem would be of form

$$\begin{aligned} \min_{(x,y)} \; & c^T x \;+\; (d + Dq)^T y \;+\; \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^T \left(\begin{bmatrix} Q_1 & R \\ R^T & Q_2 \end{bmatrix} + F(D) \right) \begin{pmatrix} x \\ y \end{pmatrix} \\ \text{s.t.} \;\; & Ax \;+\; By \;=\; f \\ & 0 \;\le\; y \;\perp\; w = q \;+\; Nx \;+\; My \;\ge\; 0. \end{aligned} \tag{3}$$

QPCC (3) is totally equivalent to QPCC (1) due to the complementarity relationships between nonnegative variables y and w. A suitable nonnegative diagonal matrix D can be picked by solving an SDP

$$\begin{aligned} \max_{(d_1,\cdots,d_m)} \;\; & \textstyle\sum_{i=1}^m d_i \, p_i \\ \text{s.t.} \qquad & -F(D) \preceq \begin{bmatrix} Q_1 & R \\ R^T & Q_2 \end{bmatrix} \\ & d_i \;\ge\; 0, \; \forall i. \end{aligned} \tag{4}$$

One possible choice of p_i could be $\bar{w}_i \bar{y}_i$ with $(\bar{x}, \bar{y}, \bar{w})$ being an initial infeasible point of the QPCC. This choice is investigated in the computational results in §5.

3 Extending quadratic convex reformulation to convex QPCCs

The quadratic convex reformulation (QCR) technique of Billionnet et al. [1,2] (see also Galli and Letchford [10]) exploits Lagrangian duality to modify an integer program into an equivalent problem with a tighter convex relaxation. A reformulation of the objective function can be constructed using the dual optimal solution of an SDP lifting of the original binary quadratic program. The reformulated quadratic program then has a convex quadratic objective function and the tightest convex continuous relaxation. The technique was designed to work with either binary quadratic programs, or general integer quadratic programs. In the latter case, the problem is first reformulated as an equivalent binary quadratic program.

The restriction that a variable z be binary can be represented equivalently using the quadratic equality constraint $z(1-z)=0$, so a quadratic integer program can be represented as a quadratically constrained quadratic program (QCQP). A QPCC is also an example of a QCQP, so we work in the more general framework of QCQPs. Consider the QCQP

$$\begin{aligned} \min_x \;& q_0(x) := x^T Q_0 x + c_0^T x + d_0 \\ \text{s.t.} \;& Ax = b \\ & x \geq 0, \quad j \in J \subseteq \{1,\ldots,n\} \\ & q_i(x) := x^T Q_i x + c_i^T x + d_i = 0 \quad i = 1,\ldots,m \end{aligned} \tag{5}$$

where $A \in I\!R^{k\times n}$ and all other matrices and vectors are dimensioned appropriately. If a QCQP has quadratic inequality constraints, slack variables can be included explicitly to give a problem in the form (5). The quadratic constraints may include some or all of the constraints implied by the linear constraints, thus:

$$x_i(Ax-b)_j = 0 \quad \forall i,j, \quad x_j x_k \geq 0 \quad \forall j,k \in J, \quad (Ax-b)_j(Ax-b)_k = 0 \quad \forall j,k.$$

The reformulated quadratic objective function will have the form

$$q_\mu(x) \;:=\; x^T Q_0 x + c_0^T x + d_0 + \sum_{i=1}^m \mu_i (x^T Q_i x + c_i^T x + d_i) \tag{6}$$

so $q_\mu(x) = q_0(x)$ for any x feasible in (5). We denote

$$\Phi := \{(x,y) \mid Ax = b,\, x_j \geq 0 \;\forall j \in J\}, \tag{7}$$

the feasible set of the QP relaxation of (5), and we use x_J to denote the components of x in the set J. We also define

$$Q_\mu := Q_0 + \sum_{i=1}^m \mu_i Q_i \tag{8}$$

$$c_\mu := c_0 + \sum_{i=1}^m \mu_i c_i \tag{9}$$

$$d_\mu := d_0 + \sum_{i=1}^m \mu_i d_i \tag{10}$$

respectively the matrix of the quadratic term, the linear term, and the constant term in the reformulated objective function. This matrix Q_μ needs to be positive semidefinite for the relaxation to be a convex QP. The quadratic reformulation of (5) is

$$\begin{aligned} \min_x \ & x^T Q_\mu x + c_\mu^T x + d_\mu \\ \text{s.t.} \ & Ax = b \\ & x \geq 0, \quad j \in J \subseteq \{1, \ldots, n\} \\ & q_i(x) := x^T Q_i x + c_i^T x + d_i = 0 \quad i = 1, \ldots, m \end{aligned} \tag{11}$$

and the corresponding quadratic convex reformulation is

$$\begin{aligned} \min_x \ & x^T Q_\mu x + c_\mu^T x + d_\mu \\ \text{s.t.} \ & x \in \Phi. \end{aligned} \tag{12}$$

By construction, every feasible solution to (5) is feasible in (12), and has the same value in both problems. In order to have the QP relaxation bound as tight as possible, the following max-min problem needs to be solved:

$$\begin{aligned} & \max_{Q_\mu \succeq 0} \min_{x \in \Phi} q_\mu(x) \\ \equiv & \max_{Q_\mu \succeq 0} \min_x \{q_\mu(x) \mid Ax = b, x_J \geq 0\} \\ \equiv & \max_{\begin{pmatrix} Q_\mu \succeq 0 \\ (\beta, \lambda_J) \in (\mathbb{R}^k \times \mathbb{R}_+^{|J|}) \end{pmatrix}} \min_x q_\mu(x) + \beta^T(Ax - b) - \lambda_J^T x_J \\ \equiv & \max_{\begin{pmatrix} Q_\mu \succeq 0 \\ (\beta, \lambda_J) \in (\mathbb{R}^k \times \mathbb{R}_+^{|J|}) \end{pmatrix}} \min_x q_0(x) + \sum_{i=1}^m \mu_i q_i(x) + \beta^T(Ax - b) - \lambda_J^T x_J. \end{aligned} \tag{13}$$

Note that we maintain the convexity of the quadratic function $q_\mu(x)$ by requiring $Q_\mu \succeq 0$; the second equation holds because of the fact that there is no duality gap in a linearly constrained convex quadratic program. Max-min problem (13) is the Lagrangian dual of the quadratically constrained quadratic program (5), since the matrix Q_μ must be positive semidefinite for the inner minimization problem to have finite optimal value. Therefore, we reduce the problem of finding the tightest QP relaxation to solving a max-min problem, which is the Lagrangian dual of a QCQP. We summarize this in the following lemma.

Lemma 1 *The value of the tightest quadratic convex reformulation of the form* (12) *of the QCQP (5) is equal to the optimal value of its Lagrangian dual.*

The Lagrangian dual problem is equivalent to a semidefinite program, as we now show (see Fujie and Kojima [5], Lemaréchal and Oustry [11,12], Poljak, Rendl, and Wolkowicz [15], and Galli and Letchford [10] for variants of these results). The Lagrangian dual function is

$$\Theta(\mu, \beta, \lambda) := \min_x q_0(x) + \sum_{i=1}^m \mu_i q_i(x) + \beta^T(Ax - b) - \lambda_J^T x_J \tag{14}$$

with $\mu \in \Re^m$, $\beta \in \Re^k$, $\lambda \in \Re^n$, and $\lambda_j = 0$ for $j \notin J$. Let $\nu := (\mu, \beta, \lambda)$ and define

$$Q(\nu) := Q_0 + \sum_{i=1}^{m} \mu_i Q_i \;=\; Q_\mu \tag{15}$$

$$c(\nu) := c_0 + A^T\beta - \lambda + \sum_{i=1}^{m} \mu_i c_i \;=\; c_\mu + A^T\beta - \lambda \tag{16}$$

$$d(\nu) := d_0 - b^T\beta + \sum_{i=1}^{m} \mu_i d_i \;=\; d_\mu - b^T\beta, \tag{17}$$

so

$$\Theta(\nu) \;:=\; \min_x \left(x^T Q(\nu) x + c(\nu)^T x + d(\nu)\right). \tag{18}$$

Let Γ denote the set of vectors ν with $\lambda \geq 0$. The Lagrangian dual problem can be stated as

$$\max_{\nu \in \Gamma} \Theta(\nu) \tag{19}$$

or equivalently

$$\begin{array}{ll} \max\limits_{\nu \in \Gamma, r} & r \\ \text{s.t.} & x^T Q(\nu) x + c(\nu)^T x + d(\nu) \geq r \quad \forall x \end{array} \tag{20}$$

or equivalently

$$\begin{array}{ll} \max\limits_{\nu \in \Gamma, r} & r \\ \text{s.t.} & \begin{pmatrix} 1 \\ x \end{pmatrix}^T \begin{pmatrix} d(\nu) - r & \frac{1}{2}c(\nu)^T \\ \frac{1}{2}c(\nu) & Q(\nu) \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \geq 0 \quad \forall x \end{array} \tag{21}$$

or equivalently

$$\begin{array}{ll} \max\limits_{\nu \in \Gamma, r} & r \\ \text{s.t.} & \begin{pmatrix} d(\nu) - r & \frac{1}{2}c(\nu)^T \\ \frac{1}{2}c(\nu) & Q(\nu) \end{pmatrix} \succeq 0. \end{array} \tag{22}$$

It is clear that any feasible ν and r for (22) is feasible in (21). We show the converse in the following lemma, using a slightly more general notation with a contrapositive argument.

Lemma 2 *Assume the square symmetric matrix*

$$\bar{M} := \begin{pmatrix} \zeta & \beta^T \\ \beta & M \end{pmatrix}$$

is not positive semidefinite, where ζ is a scalar and β a vector. Then there exists a vector x with

$$\begin{pmatrix} 1 \\ x \end{pmatrix}^T \bar{M} \begin{pmatrix} 1 \\ x \end{pmatrix} \;<\; 0.$$

Proof Let $(v_0, v^T)^T$ satisfy $(v_0, v^T)\bar{M}(v_0, v^T)^T < 0$. If $v_0 \neq 0$ then rescaling the vector gives an appropriate vector $x = v/v_0$. If $v_0 = 0$ then $(1, \delta v^T)\bar{M}(1, \delta v^T)^T < 0$ for sufficiently large δ.

This equivalence between QCR and Lagrangian duality is summarized in the following theorem that extends Lemma 1.

Theorem 1 *The value of the tightest quadratic convex reformulation of the form* (12) *of the QCQP (5) is equal to the optimal value of its Lagrangian dual, and is equal to the value of the semidefinite program (22).*

4 Relating the value of the QCR to the value of an SDP lifting

Consider the QCQP (5). An SDP lifting of this problem is:

$$
\begin{aligned}
\min_{x,X}\ & q_0(x,X) := < Q_0, X > + c_0^T x + d_0 \\
\text{s.t.}\ & Ax = b \\
& x_j \geq 0, \quad j \in J \subseteq \{1, \ldots, n\} \\
& q_i(x,X) := < Q_i, X > + c_i^T x + d_i = 0, \quad i = 1, \ldots, m \\
& \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0.
\end{aligned}
\tag{23}
$$

This is a convex relaxation of the original QCQP, so its optimal value provides a lower bound on the optimal value of the QCQP. Further, its dual is the problem (22). Therefore, we have the following theorem.

Theorem 2 *The optimal value of the Lagrangian dual of the QCQP is equal to the optimal value of the dual of the standard SDP lifting of the QCQP. If strong duality holds for the SDP then the optimal value of the Lagrangian dual of the QCQP is equal to the optimal value of its SDP relaxation.*

The Slater constraint qualification for the SDP pair holds if the objective function matrix Q in (1) is positive definite. Thus we have the following corollary.

Corollary 1 *When the Q matrix in QPCC* (1) *is positive definite, constraint qualification holds for the SDP pair (22) and (23), so the optimal value of the convex quadratic reformulation is equal to the value of the SDP relaxation of the QPCC.*

Let us assume (22) has an optimal solution, denoted by $\nu^* = (\mu^*, \beta^*, \lambda^*)$. Note that Q_{μ^*} is positive semidefinite, so problem (12) with $\mu = \mu^*$ is a convex relaxation of our QCQP. The advantage of this formulation over (22) is that it is in the space of the original variables. It is a convex formulation, and its value is equal to the optimal value of its Lagrangian dual, and also to the SDP relaxation if strong duality holds. It can be employed in a branch-and-cut framework to solve the original problem. Convex relaxations of the quadratic

Table 1 $[(m, n, k) = (50, 10, 5)]$ gap closed

	opt	QP_{rlx}	lb($y^T Dw$)	% gap closed	lb (QCR)	% gap closed	(penalty) time(s)	(SDPrlx) time(s)
1	-309.8868	-310.2951	-309.9126	93.68	-309.8871	99.93	1.19	3.78
2	-67.4262	-67.4938	-67.431	92.90	-67.4278	97.63	1.13	2.94
3	-138.8104	-139.6244	-138.8375	96.67	-138.8175	99.13	1.11	4.18
4	-143.4102	-143.4689	-143.4115	97.79	-143.4102	100.00	1.24	3.40
5	-264.4806	-264.5879	-264.4823	98.42	-264.4806	100.00	1.16	2.56
6	-145.4621	-145.5447	-145.4703	90.07	-145.4621	100.00	1.12	2.75
7	-20.9007	-21.7369	-21.1698	67.82	-21.1114	74.80	1.15	2.84
8	-197.8096	-197.8545	-197.8098	99.55	-197.8096	100.00	1.21	2.92
9	-217.5868	-217.6531	-217.5894	96.08	-217.5872	99.40	1.20	2.31
10	-251.3961	-251.4479	-251.4017	89.19	-251.3966	99.03	1.21	2.85
				92.22		96.99	1.17	3.15

constraints in (5) can be incorporated into (12) to tighten it and subsequent relaxations in a branch-and-cut framework.

Formulation (12) is a valid convex relaxation of the QCQP for any ν^* that is feasible in (22); it is not necessary that ν^* be optimal. For example, it is not necessary to include all the original constraints in the Lagrangian, allowing a trade-off between speed of solving the semidefinite program and the quality of the bound obtained from the relaxation.

5 Some Computational Experiments

In this section, we present some computational results in Tables 1, 2, 3 and 4. The experiments were run on QPCC problems with strictly convex objective functions, which are a special family of QCQPs. We require the strict convexity of the quadratic objective function so that the strong duality holds for the SDP lifting. The computational results show that the reformulated QPCC has a tighter QP relaxation bound. The computational experiments were run on QPCCs with equality side constraints $Ax + By = f$ (Tables 1, 2 and 3), or with inequality side constraints $Ax + By \geq f$ (Table 4). In addition to the complementarity restrictions, the nonconvex quadratic constraints $x_i(Ax + By - f)_j = 0$, $i = 1, \ldots, n$, $j = 1, \ldots, k$ were included in the Lagrangian relaxation, for the problems with equality side constraints. The example QPCCs were generated following the QPECgen method [9]. As a stationary point is generated in advance, we have an upper bound on the example QPCC. For Tables 1, 2 and 4, we could solve the example QPCCs using the method proposed in [3], therefore we use the optimal values as the upper bounds.

In each of the tables, the column of "lb($y^T Dw$)" contains the QP relaxation bounds we obtained using the scheme (4) of adding $y^T Dw$ to the QPCC objective functions, while the column of "lb(QCR)" contains the QP relaxation bounds we obtained using the QCR method. Also, the percentage of gap

Table 2 $[(m, n, k) = (100, 10, 2)]$ gap closed

	opt	QP_{rlx}	lb($y^T Dw$)	% gap closed	lb (QCR)	% gap closed	(penalty) time(s)	(SDPrlx) time(s)
1	-6.0442	-9.236	-7.4163	57.01	-7.2976	60.73	2.83	9.57
2	-30.2126	-30.5237	-30.2514	87.53	-30.2216	97.11	3.28	10.29
3	-46.1831	-46.6227	-46.2822	77.46	-46.2076	94.43	2.87	9.60
4	-6.9413	-8.7065	-7.4089	73.51	-7.2808	80.77	2.70	8.91
5	-22.5016	-23.41	-22.7475	72.93	-22.6393	84.84	2.60	9.00
6	-41.5942	-41.7507	-41.6029	94.44	-41.5947	99.68	2.92	11.94
7	-106.4055	-106.5383	-106.4105	96.23	-106.406	99.77	2.61	11.27
8	-25.0137	-25.4995	-25.1382	74.37	-25.0999	82.26	2.90	10.48
9	-2.2911	-4.6037	-3.1505	62.84	-2.9326	72.26	2.47	6.98
				77.51		85.92	2.77	9.79

Table 3 $[(m, n, k) = (300, 10, 5)]$ gap closed

	ub	QP_{rlx}	lb($y^T Dw$)	% gap closed	lb (QCR)	% gap closed	(penalty) time(s)	(SDPrlx) time(s)
1	-101.6011	-102.1054	-101.7406	72.34	-101.7021	79.97	93.99	192.38
2	-62.5099	-68.4999	-64.2666	70.67	-63.7324	79.59	69.27	166.23
3	-5.2152	-9.7033	-7.0351	59.45	-6.6766	67.44	67.44	188.13
4	-2.5884	-29.8765	-25.8613	14.71	-25.7312	15.19	77.11	178.86
5	-256.6023	-258.1193	-256.7252	91.90	-256.7252	91.90	74.18	74.18
6	-6.0225	-9.7167	-7.3488	64.10	-6.982	74.03	79.37	174.45
7	-13.2958	-20.2235	-17.0647	45.60	-16.8887	48.14	75.10	147.78
8	-21.8755	-26.3378	-23.3456	67.06	-22.9582	75.74	72.86	179.06
9	-58.8571	-60.7365	-59.2903	76.95	-59.1005	87.05	83.19	178.55
10	-2.9081	-8.4514	-5.7658	48.45	-5.5232	52.82	69.80	156.72
				61.12		67.19	69.20	163.63

Table 4 $[(m, n, k) = (90, 5, 10)]$ gap closed

	opt	QP_{rlx}	lb($y^T Dw$)	% gap closed	lb (QCR)	% gap closed	(penalty) time(s)	(SDPrlx) time(s)
1	2.3344	-34.2936	1.1401	96.74	2.2081	99.66	2.32	8.40
2	-1.2054	-22.5043	-4.1704	86.08	-2.605	93.43	2.27	7.92
3	-1.2328	-4.2513	-2.6524	52.97	-2.4093	61.02	2.59	8.19
4	-0.4906	-2.1556	-0.49067	100.00	-0.4906	100.00	3.14	6.73
5	87.8006	-10.926	87.8006	100.00	87.8006	100.00	2.71	7.05
6	-22.2627	-130.2954	-22.2708	99.99	-22.2638	100.00	2.04	8.51
7	-37.6909	-63.722	-37.6909	100.00	-37.6909	100.00	2.70	7.22
8	-34.9749	-45.2988	-35.1912	97.90	-34.9794	99.96	2.03	10.49
9	-1.9693	-27.9661	-6.4096	82.92	-3.8904	92.61	2.53	7.74
10	0.5298	-13.6261	-1.9429	82.53	-0.0577	95.85	2.63	8.33
11	-0.3549	-3.7196	-0.3667	99.65	-0.3601	99.85	2.42	7.80
12	0.9751	-51.1095	-17.0688	65.36	-11.2974	76.44	2.81	8.29
				88.68		93.23	2.50	8.13

closed is defined as

$$\frac{lb - QP_{rlx}}{ub - QP_{rlx}} \times 100\%,$$

where lb is the new QP relaxation bound after reformulation (lb($y^T Dw$) or lb(QCR)), QP_{rlx} is the original QP relaxation bound and ub is the upper bound on the QPCC.

The QPCCs in Tables 1, 2 and 3 are subject to equality side constraints $Ax + By = f$ and linear complementarity constraints. On average, the QP relaxation bounds by the method of adding $y^T Dw$ to the QPCC objective functions can close 92.22%, 77.51% and 61.12% of the gaps respectively; whereas the QP relaxation bounds by the QCR method can close 96.99%, 85.92% and 67.19% of the gaps respectively. The QPCCs in Table 4 are subject to inequality side constraints $Ax + By \geq f$ and linear complementarity constraints. By the method (4) of adding $y^T Dw$ to the QPCC objective functions, the QP relaxation bounds can close 88.68% of the gaps on average; whereas by the QCR method, the QP relaxation bounds can close 93.23% of the gaps on average.

6 Concluding Remarks

The Quadratic Convex Reformulation (QCR) method, proposed by [1,2], is a technique for nonconvex binary quadratic programs. The present paper extends the QCR method to general QCQPs, in particular to QPCC problems with convex objective functions. We perturb the quadratic objective function of the QPCC using the dual optimal solution of an SDP relaxation problem in such a manner that (1) the convexity of the objective function as well as the global resolution of the QPCC are maintained; (2) the QP relaxation of the reformulated QPCC provides a lower bound that is as tight as possible. Aside from getting a tight lower bound, this QPCC reformulation technique could also be used as a preprocessing step when solving the QPCC, as the scheme of adding $y^T Dw$ to the QPCC objective function is used in [3].

We also extend the QCR method to general quadratically constrained quadratic programs. In general, we use the dual optimal solution of the QCQP's SDP lifting to perturb the objective function so that the QP relaxation bound is as tight as possible. When strong duality holds for the SDP lifting, the QP relaxation of the perturbed QCQP is as tight as the SDP lifting. However, the advantage of the QP relaxation is that it is in the space of the original variables; and the perturbation could be constructed using a feasible dual solution instead of the optimal dual solution when the SDP is hard to solve.

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